

Error analysis for a fractional Adams method

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ABSTRACT. This report, we investigate a method for the numerical solution of the nonlinear fractional differential equation $D_*^\alpha y(t) = f(t, y(t))$, equipped with initial conditions $y^{(k)}(0) = y_0^{(k)}$, $k = 0, 1, \dots, [\alpha] - 1$; Here α may be an arbitrary positive real number, and the differential operator is the Caputo derivative. In this report, firstly the numerical method (the Adams method) is given, then we further study the error analysis of this method for the fractional differential equations in the cases: (i) the error estimates for $\alpha \in (0, 1)$, $f \in C^2(G)$, (ii) the solution $y(t)$ having some special forms

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1. Introduction. The generalized Adams-Bashforth-Moulton method, often simply called "the fractional Adams method", is a useful numerical algorithm for solving a fractional ordinary differential equation: $D_*^\alpha y(t) = f(t, y(t))$, $y^{(k)}(0) = y_0^{(k)}$, $k = 0, 1, \dots, n - 1$, where $\alpha > 0$, $n = [\alpha]$ is the first integer not less than α , and $D_*^\alpha y(t)$ is the α th-order fractional derivative of $y(t)$ in the Caputo sense. Although error analyses for this fractional Adams method have been given for (a) $0 < \alpha$, $D_*^\alpha y(t) \in C^2[0, T]$, (b) $\alpha > 1$, $y \in C^{1+[\alpha]}[0, T]$, (c) $0 < \alpha < 1$, $y \in C^2[0, T]$, (d) $\alpha > 1$, $f \in C^3(G)$, there are still some unsolved problems (i) the error estimates for $\alpha \in (0, 1)$, $f \in C^2(G)$, (ii) the solution $y(t)$ having some special forms. In this report, we mainly study the error analyses of the fractional Adams method for the fractional ordinary differential equations for the three cases (i)-(ii).

2. The Adams method

Let the fractional initial value problem:

$$\begin{cases} D_*^\alpha y(t) = f(t, y(t)) \\ y^{(k)}(0) = y_0^{(k)}, k = 0, 1, \dots, n - 1 \end{cases} \quad (1)$$

where the $y_0^{(k)}$ may be arbitrary real numbers, $\alpha > 0$ and $n = [\alpha]$ is the smallest integer $\geq \alpha$. $D_*^\alpha y(t)$ is the Caputo fractional derivative of order α of function $y(t)$, which is defined by

$$D_*^\alpha z(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n - \alpha - 1} z^{(n)}(\tau) d\tau \quad (0 \leq n - 1 < \alpha < n \in \mathbb{Z}^+)$$

In this part, the our purpose is to give a numerical method for (1)- the fractional Adams-Bashforth-Moulton method, simply called "the fractional Adams method". Firstly, we have the initial value problem (1) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - u)^{\alpha-1} f(u, y(u)) du \quad (2)$$

in the sense that a continuous function is a solution of (1) if and only if it is a solution of (2).

Here, we assume the function f to be such that a unique solution exists on some interval $[0, T]$. we also assume that we are working on a uniform grid $\{t_j = jh: j = 0, 1, \dots, N\}$ with some integer N and $h = T/N$. The basic idea is, assuming that we have already calculated approximations $y_j \approx y(t_j)$ ($j = 1, 2, \dots, k$), that we try to obtain the approximation y_{k+1} by means of the equation

$$y(t_{k+1}) = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - u)^{\alpha-1} f(u, y(u)) du \quad (3)$$

To construct the required formula, we use the product trapezoidal quadrature formula with respect to the

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weight function $(t_{k+1} - \cdot)^{\alpha-1}$ to replace the integral. In other words, we apply the approximation

$$\begin{aligned} \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} f(t, y(t)) dt &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1} - t)^{\alpha-1} f(t, y(t)) dt \\ &\approx \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1} - t)^{\alpha-1} \left[f(t_j, y(t_j)) \right. \\ &\quad \left. + \frac{1}{h} (f(t_{j+1}, y(t_{j+1})) - f(t_j, y(t_j))) (t - t_j) \right] dt \end{aligned} \quad (4)$$

where, by simple calculations we have

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \times \begin{cases} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha) & \text{if } j = 0 \\ ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1}) & \text{if } 1 \leq j \leq k \\ -2(k-j+1)^{\alpha+1} & \text{if } j = k+1 \\ 1 & \text{if } j = k+1 \end{cases} \quad (5)$$

This then gives us our corrector formula (i.e., the fractional variant of the one-step Adams-Moulton method), which is

$$y_{k+1} = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^p) \right) \quad (6)$$

The remaining problem is the determination of the predictor formula required to calculate y_{k+1}^p . To do this, we replace the integral on the right-hand side of equation (3) by the product rectangle rule

$$\begin{aligned} \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} f(t, y(t)) dt &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1} - t)^{\alpha-1} f(t, y(t)) dt \\ &= \sum_{j=0}^k b_{j,k+1} f(t_j, y(t_j)) \end{aligned}$$

where now

$$b_{j,k+1} = \frac{h^\alpha}{\alpha} ((k+1-j)^\alpha - (k-j)^\alpha) \quad (8)$$

Thus, the predictor y_{k+1}^p is determined by the fractional Adams-Bashforth method

$$y_{k+1}^p = \sum_{k=0}^{n-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k b_{j,k+1} f(t_j, y_j) \quad (9)$$

Finally, we have the fractional Adams-Bashforth-Moulton method, which is written as following

$$\begin{cases} y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^p) \right) \\ y_{k+1}^p = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k b_{j,k+1} f(t_j, y_j) \end{cases} \quad (10)$$

in which, the weights $a_{j,k+1}$ and $b_{j,k+1}$ being defined according to (5) and (8), respectively.

We have thus completed the description of our numerical algorithm. The remainder of this report will be devoted to the error analysis for this scheme.

3. Error analyses for the fractional Adams method

In this section, we give and prove theorems below.

Lemma. If $\psi \in C^1[0, T]$, $0 < \alpha < 1$, then

$$E_{k+1} \left[\int_0^t (t - \tau)^{-\alpha} \psi'(\tau) d\tau \right] \leq \frac{2}{\alpha(1-\alpha)} \|\psi'\| t_{k+1}^\alpha h^{1-\alpha}$$

and

$$F_{k+1} \left[\int_0^t (t - \tau)^{-\alpha} \psi'(\tau) d\tau \right] \leq \frac{A}{1-\alpha} \|\psi'\|_\infty t_{k+1}^\alpha h^{1-\alpha}$$

where A is a constant independent of k and h .

Proof. Using the expressions for E_{k+1} , we have

$$\begin{aligned}
E_{k+1} \left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] &= \\
&= \left| \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \left\{ \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right\} (t_{k+1}-t)^{\alpha-1} dt \right| \\
&\leq \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \left| \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right| (t_{k+1}-t)^{\alpha-1} dt \\
&\quad \left| \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right| \\
&\leq \|\psi'\|_{\infty} \left(\int_0^{t_j} (t_j-\tau)^{-\alpha} d\tau - \int_0^{t_j} (t-\tau)^{-\alpha} d\tau + \int_{t_j}^t (t-\tau)^{-\alpha} d\tau \right) \\
&\leq \frac{1}{1-\alpha} \|\psi'\|_{\infty} (t_j^{1-\alpha} + (t-t_j)^{1-\alpha} - t^{1-\alpha} + (t-t_j)^{1-\alpha}) \\
&\leq \frac{2}{1-\alpha} \|\psi'\|_{\infty} (t-t_j)^{1-\alpha} \leq \frac{2}{1-\alpha} \|\psi'\|_{\infty} h^{1-\alpha}.
\end{aligned}$$

So, one has

$$\begin{aligned}
E_{k+1} \left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] &\leq \frac{2}{1-\alpha} \|\psi'\|_{\infty} h^{1-\alpha} \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} dt \\
&= \frac{2}{1-\alpha} \|\psi'\|_{\infty} h^{1-\alpha} \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} dt = \frac{2}{\alpha(1-\alpha)} \|\psi'\|_{\infty} t_{k+1}^{\alpha} h^{1-\alpha}
\end{aligned}$$

Applying the expression for F_{k+1} gives

$$\begin{aligned}
F_{k+1} \left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] &= \\
&= \left| \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} \left\{ \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right. \right. \\
&\quad \left. \left. - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau - \frac{1}{h} \left[\int_0^{t_{j+1}} (t_{j+1}-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right] (t-t_j) \right\} dt \right| \\
&\leq \left| \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} \left| \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right. \right. \\
&\quad \left. \left. - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau - \frac{1}{h} \left[\int_0^{t_{j+1}} (t_{j+1}-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right] (t-t_j) \right| dt \right|.
\end{aligned}$$

By tedious calculations, one gets

$$\begin{aligned}
&\left| \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right. \\
&\quad \left. - \frac{1}{h} \left[\int_0^{t_{j+1}} (t_{j+1}-\tau)^{-\alpha} \psi'(\tau) d\tau - \int_0^{t_j} (t_j-\tau)^{-\alpha} \psi'(\tau) d\tau \right] (t-t_j) \right| \\
&\leq \left| \int_0^{t_j} \left[(t-\tau)^{-\alpha} - (t_j-\tau)^{-\alpha} - \frac{1}{h} ((t_{j+1}-\tau)^{-\alpha} - (t_j-\tau)^{-\alpha})(t-t_j) \right] \psi'(\tau) d\tau \right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_{t_j}^t \left[(t-\tau)^{-\alpha} - \frac{1}{h} (t_{j+1}-\tau)^{-\alpha} (t-t_j) \right] \psi'(\tau) d\tau \right| + \left| \frac{1}{h} \int_t^{t_{j+1}} (t_{j+1}-\tau)^{-\alpha} (t-t_j) \psi'(\tau) d\tau \right| \\
& \leq \frac{1}{1-\alpha} \|\psi'\|_\infty \left\{ \left| -(t-t_j)^{1-\alpha} + t^{1-\alpha} - t_j^{1-\alpha} - \frac{1}{h} (t-t_j) \left[-(t_{j+1}-t_j)^{1-\alpha} + t_{j+1}^{1-\alpha} - t_j^{1-\alpha} \right] \right| \right. \\
& \quad \left. + \left| (t-t_j)^{1-\alpha} - \frac{1}{h} (t-t_j) \left[-(t_{j+1}-t)^{1-\alpha} + (t_{j+1}-t_j)^{1-\alpha} \right] \right| + \frac{1}{h} (t_{j+1}-t)^{1-\alpha} (t-t_j) \right\} \\
& \leq \frac{1}{1-\alpha} \|\psi'\|_\infty \left\{ t^{1-\alpha} - t_j^{1-\alpha} - \frac{1}{h} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) (t-t_j) \right. \\
& \quad \left. + \left((t-t_j)^{1-\alpha} - h^{-\alpha} (t-t_j) \right) + (t-t_j)^{1-\alpha} - h^{-\alpha} (t-t_j) + \frac{2}{h} (t_{j+1}-t)^{1-\alpha} (t-t_j) \right\} \\
& = \frac{1}{1-\alpha} \|\psi'\|_\infty \left\{ t^{1-\alpha} - t_j^{1-\alpha} - \frac{1}{h} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) (t-t_j) \right. \\
& \quad \left. + 2 \left[h^{-1} (t_{j+1}-t)^{1-\alpha} (t-t_j) + (t-t_j)^{1-\alpha} - h^{-\alpha} (t-t_j) \right] \right\} \\
& \leq \frac{1}{1-\alpha} \|\psi'\|_\infty \left\{ t^{1-\alpha} - t_j^{1-\alpha} - \frac{1}{h} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) (t-t_j) + 4h^{1-\alpha} \right\}.
\end{aligned}$$

Utilizing the above inequalities, we have

$$\begin{aligned}
F_{k+1} \left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] & \leq \frac{1}{1-\alpha} \|\psi'\|_\infty \left\{ F_{k+1}[t^{1-\alpha}] + 4h^{1-\alpha} \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1}-t)^{\alpha-1} dt \right\} \\
& \leq \frac{1}{1-\alpha} \|\psi'\|_\infty \left\{ C_{\alpha,1-\alpha}^{\text{Tr}} t_{k+1}^{\alpha-1} h^{2-\alpha} + \frac{4}{\alpha} t_{k+1}^\alpha h^{1-\alpha} \right\} \leq \frac{A}{1-\alpha} \|\psi'\|_\infty t_{k+1}^\alpha h^{1-\alpha}
\end{aligned}$$

The proof is completed.

Theorem 1. Let $0 < \alpha < 1$. Then, if $f \in C^2(G)$,

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = O(h^{1-\alpha})$$

for every $\epsilon > 0$, and

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = O(1)$$

Proof. Since $f \in C^2(G)$, using Theorem 2.1(a) in [K. Diethelm, N.J. Ford, A.D. Freed, 2004] we have

$$y(t) = \sum_{v=1}^{v_1} c_v t^{\alpha v} + \psi(t)$$

in which $v_1 = \lceil 1/\alpha \rceil - 1$, $\psi(t) \in C^1[0, T]$.

Now, using the Caputo fractional derivative definition yields

$$D_*^\alpha y(t) = \sum_{v=1}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} c_v t^{\alpha v - \alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \quad (11)$$

Set $g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau$. Next we estimate $E_{k+1}[g(t)]$ and $F_{k+1}[g(t)]$.

$$\begin{aligned}
E_{k+1}[g(t)] & = \frac{1}{\Gamma(1-\alpha)} E_{k+1} \left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] \\
& \leq \frac{1}{\Gamma(1-\alpha)} \frac{2}{\alpha(1-\alpha)} \|\psi'\| t_{k+1}^\alpha h^{1-\alpha}
\end{aligned} \quad (12)$$

Also,

$$\begin{aligned}
F_{k+1}[g(t)] & = \frac{1}{\Gamma(1-\alpha)} F_{k+1} \left[\int_0^t (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] \\
& \leq \frac{A}{\Gamma(2-\alpha)} \|\psi'\|_\infty t_{k+1}^\alpha h^{1-\alpha}
\end{aligned} \quad (13)$$

From Theorem 2.4(b) in [K. Diethelm, N.J. Ford, A.D. Freed, 2004] we have

$$\begin{aligned}
E_{k+1} \left[\sum_{v=1}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} c_v t^{\alpha v - \alpha} \right] & \leq \sum_{v=1}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} |c_v| E_{k+1}[t^{\alpha v - \alpha}] \\
& \leq \sum_{v=1}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} |c_v| C_{\alpha, \alpha v - \alpha}^{\text{Re}} t_{k+1}^{\alpha + \alpha v - \alpha - 1} h \leq C^{\text{Re}} t_{k+1}^{\alpha-1} h
\end{aligned} \quad (14)$$

and from Theorem 2.4(c) in [K. Diethelm, N.J. Ford, A.D. Freed, 2004], we have

$$\begin{aligned} F_{k+1} \left[\sum_{v=1}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} c_v t^{\alpha v - \alpha} \right] &\leq \sum_{v=2}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} |c_v| F_{k+1} [t^{\alpha v - \alpha}] \\ &\leq \sum_{v=2}^{v_1} \frac{\Gamma(\alpha v + 1)}{\Gamma(\alpha v + 1 - \alpha)} |c_v| C_{a, \alpha v - \alpha}^{\text{Tr}} t_{k+1}^{\alpha + \alpha v - \alpha - \sigma_v} h^{\sigma_v} \leq C^{\text{Tr}} t_{k+1}^{\alpha - 1} h^{\alpha + 1} \end{aligned} \quad (15)$$

where $\sigma_v = \min(2, \alpha v - \alpha + 1)$.

Thus, from (11), (12) and (14) we get

$$E_{k+1} [D_*^\alpha y(t)] \leq 2 \max \left\{ \frac{2}{\alpha \Gamma(2 - \alpha)} \|\psi'\| t_{k+1}^\alpha h^{1 - \alpha}, C^{\text{Re}} t_{k+1}^{\alpha - 1} h \right\} \quad (16)$$

and from (11), (13) and (15) we also get

$$F_{k+1} [D_*^\alpha y(t)] \leq 2 \max \left\{ \frac{A}{\Gamma(2 - \alpha)} \|\psi'\| t_{k+1}^\alpha h^{1 - \alpha}, C^{\text{Tr}} t_{k+1}^{\alpha - 1} h^{\alpha + 1} \right\} \quad (17)$$

Now, by using (16) and (17), we obtain

$$|y(t_j) - y_j| \leq C t_j^{\alpha - 1} h^q, \quad (q = \min(1, 1 - \alpha))$$

So, if $\epsilon \in (0, T]$ and $t_j \in [\epsilon, T]$, then

$$|y(t_j) - y_j| \leq C t_j^{\alpha - 1} h^{1 - \alpha} \leq C \epsilon^{\alpha - 1} h^q$$

or

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = O(h^{1 - \alpha})$$

and

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = O(1)$$

The proof is completed.

According to Theorems 2.1, 3.2 in [K. Diethelm, N.J. Ford, A.D. Freed, 2004], the smooth condition of $\psi(t)$ plays an important role in error estimates. If this term is removed, what will happen? We will give an answer.

Theorem 2. Suppose that the solution y of (1) is expressed in the form

$$y(t) = \sum_{v=0}^{v_0} c_v t^{\alpha v} \quad (18)$$

in which $v_0 \in \mathbb{Z}^+$, with the constants $c_1, c_2, \dots, c_{v_0} \in \mathbb{R}$. Then, for $\epsilon \in (0, T]$,

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } 1 < \alpha \\ O(h^{1 + \alpha}) & \text{if } 0 < \alpha < 1 \end{cases}$$

and moreover,

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } 1 < \alpha \\ O(h^{2\alpha}) & \text{if } 0 < \alpha < 1 \end{cases}$$

Proof. Using (18) and the Caputo fractional derivative definition yields

$$D_*^\alpha y(t) = \sum_{v=1}^{v_0} \frac{\Gamma(v\alpha + 1)}{\Gamma(v\alpha + 1 - \alpha)} c_v t^{v\alpha - \alpha} \quad (19)$$

If $\alpha > 1$, it follows from (19) and Theorems 2.3 and 2.4 in [K. Diethelm, N.J. Ford, A.D. Freed, 2004] that

$$E_{k+1} [D_*^\alpha y(t)] \leq \sum_{v=1}^{v_0} \frac{\Gamma(v\alpha + 1)}{\Gamma(v\alpha + 1 - \alpha)} |c_v| E_{k+1} [t^{v\alpha - \alpha}] \leq C_1 t_{k+1}^\alpha h \quad (20)$$

and

$$F_{k+1} [D_*^\alpha y(t)] \leq \sum_{v=1}^{v_0} \frac{\Gamma(v\alpha + 1)}{\Gamma(v\alpha + 1 - \alpha)} |c_v| F_{k+1} [t^{v\alpha - \alpha}] \leq C_2 t_{k+1}^{2(\alpha - 1)} h^2 \quad (21)$$

Now, by using (20) and (21) and Lemma 3 in [K. Diethelm, N.J. Ford, A.D. Freed, 2004], we get

$$\begin{aligned} |y(t_j) - y_j| &\leq C h^q, \quad (q = \min(2, 1 + \alpha)) \\ \max_{0 \leq j \leq N} |y(t_j) - y_j| &= O(h^2) \end{aligned}$$

If $0 < \alpha < 1$, by almost the same reasoning, we again have

$$E_{k+1} [D_*^\alpha y(t)] \leq \sum_{v=1}^{v_0} \frac{\Gamma(v\alpha + 1)}{\Gamma(v\alpha + 1 - \alpha)} |c_v| E_{k+1} [t^{v\alpha - \alpha}] \leq C_3 t_{k+1}^{\alpha - 1} h \quad (22)$$

and

$$F_{k+1} [D_*^\alpha y(t)] \leq \sum_{v=1}^{v_0} \frac{\Gamma(v\alpha + 1)}{\Gamma(v\alpha + 1 - \alpha)} |c_v| F_{k+1} [t^{v\alpha - \alpha}] \leq C_4 t_{k+1}^{\alpha - 1} h^{\alpha + 1} \quad (23)$$

By using (22) and (23), we obtain

$|y(t_j) - y_j| \leq C_5 t^{\alpha-1} h^q$, (here $q = 1 + \alpha$)
So, for arbitrary $\epsilon > 0$ and $t_j \in [\epsilon, T]$,

$$|y(t_j) - y_j| \leq C_5 \epsilon^{\alpha-1} h^{1+\alpha}$$

i.e.,

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = O(h^{1+\alpha})$$

and for $0 \leq j \leq N$, $|y(t_j) - y_j| \leq C h^{2\alpha}$.

This ends the proof.

Compared to DFF Theorem (b) (the case with $\alpha > 1$ and $\in C^{[\alpha]+1}[0, T]$), here the order of the estimate is much higher but $y \in C^{[\alpha]-1}[0, T]$, except for the special form of $y(t)$. The error estimates are actually the posterior estimates. A more general case relating to the posterior error estimates is given below.

Theorem 3. Suppose that the solution y of (1) is expressed in the form

$$y(t) = \sum_{v=0}^{v_0} c_{0,v} t^{\alpha v} + \sum_{v=1}^{v_1} c_{1,v} t^{1+\alpha v} + \sum_{v=1}^{v_2} c_{2,v} t^{2+\alpha v} + \dots + \sum_{v=1}^{v_l} c_{l,v} t^{l+\alpha v} \quad (24)$$

in which $v_0, v_1, v_2, \dots, v_l, l \in \mathbb{Z}^+$, all $c_{0,v}, c_{1,v}, \dots, c_{l,v}$ are real numbers. Then, for $\epsilon \in (0, T]$,

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } 1 < \alpha \\ O(h^{1+\alpha}) & \text{if } 0 < \alpha < 1 \end{cases}$$

and moreover,

$$\max_{0 \leq j \leq N} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } 1 < \alpha \\ O(h^{2\alpha}) & \text{if } 0 < \alpha < 1 \end{cases}$$

The proof of this theorem is the same as that of Theorem 2 and so is omitted here. It is worthy of note that the last l terms on the right hand side of Eq. (24) cannot begin with $v = 0$; otherwise, the above conclusion does not hold yet.

Conclusion. In this report, we first introduce the fractional Adams method, then present and prove one lemma and three important theorems on the errors of this fractional Adam method. These results help improve the theory of fractional differential equations and are widely applied in other fields.

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TÓM TẮT

Phân tích sai số cho phương pháp Adams cấp phân số

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Báo cáo này, chúng tôi nghiên cứu phương pháp số để giải phương trình vi phân cấp phân số phi tuyến $D_*^\alpha y(t) = f(t, y(t))$, với các điều kiện ban đầu $y^{(k)}(0) = y_0^{(k)}$, $k = 0, 1, \dots, [\alpha] - 1$; Ở đây α có thể là một số thực dương tùy ý và toán tử vi phân là đạo hàm Caputo. Trong báo cáo này, trước hết phương pháp số (phương pháp Adams) được đưa ra, sau đó chúng tôi nghiên cứu sâu hơn về phân tích sai số của phương pháp này cho các phương trình vi phân cấp phân số trong các trường hợp: (i) ước lượng sai số trong trường hợp $\alpha \in (0, 1)$, $f \in C^2(G)$, (ii) nghiệm $y(t)$ có một số dạng đặc biệt

Từ khóa: Phép tính phân số; Đạo hàm Caputo; phương trình vi phân cấp phân số