

A highly accurate analytical method for determination of the vibrational frequency of N/MEMS with electrostatic and van der Waals interaction forces

Nguyen Nhu Hieu^{1,2,*}  and Pham Ngoc Chung³

¹ Faculty of Mechanical Engineering and Mechatronics, Phenikaa University, To Huu Str., Yen Nghia Ward, Ha Dong Dist., Hanoi, Vietnam

² Phenikaa Research and Technology Institute (PRATI), A&A Green Phoenix Group SJC, 167 Hoang Ngan Str., Trung Hoa Ward, Cau Giay Dist., Hanoi, Vietnam

³ Faculty of Basic Science, Hanoi University of Mining and Geology, 18 Vien Str., Duc Thang Ward, Bac Tu Liem Dist., Hanoi, Vietnam

E-mail: hieu.nguyennhu@phenikaa-uni.edu.vn and phamngocchung@humg.edu.vn

Received 21 June 2024, revised 18 August 2024

Accepted for publication 23 August 2024

Published 12 September 2024



Abstract

In this study, a novel approach based on the elliptic balance method (EBM) is proposed for the first time to find the approximate frequency of nano/micro-electromechanical systems modeled as Euler–Bernoulli beams under the effects of electrostatic and van der Waals interaction forces. Firstly, the governing equation of the beam is reduced to the single-mode vibration equation using the Galerkin method. A nonlinear differential equation for the time-dependent beam deflection is obtained. We present the approximate solution as an elliptic cosine function, which considers the free term contributing to the solution. This free term is relevant for vibrations with a non-zero mean in time, in which the beam is affected by a relatively large applied voltage. Via some manipulations, the obtained result is an algebraic equation with only one unknown in three unknowns: the free and vibration coefficient terms, and the modulus quantity of the elliptic cosine function. This nonlinear equation is solved using the Newton–Raphson method. The numerical results from the EBM show that the accuracy of the solution responses in time and approximate frequency is relatively accurate, almost coinciding with the results obtained from the numerical solution method using the Runge–Kutta algorithm. Our results also agree well with previously published experimental and simulation results. The results are meaningful when determining the frequency of the vibrating beam with high accuracy for micro/nano systems.

Keywords: micro/nano-beam, electrostatic force, van der Waals interaction, pull-in instability, elliptic balance method

1. Introduction

Nano/micro electromechanical systems (N/MEMSs) play an important role in nanotechnology and are commonly used in most devices containing nano/micro-sized structures such as

sensors and actuators [1–3]. The structure of the system usually has two plates acting as two electrodes, including a fixed plate, called the substrate, and a movable plate modeled as a beam so that its body and surface can be deformed and bent transversely due to the interactions forces of the beam with the fixed substrate [4, 5]. One of the problems encountered when designing or working with N/MEMS is the phenomenon of pull-in instability, as the applied DC voltage approaches a

* Author to whom any correspondence should be addressed.

threshold value at which the deflection of the beam reaches a maximum value and touches the bottom of the fixed electrode. At that time, the interaction force, such as an electrostatic force, suddenly increases, overcoming the elastic force, and can have harmful effects on the N/MEMSs. The phenomenon of pull-in instability was first observed experimentally by Nathansom *et al* [6] and Taylor [7] in 1967 and 1968, respectively. This phenomenon is often harmful to the structure, and therefore an important design task is to find the voltage point that causes the instability to the system. This is an interesting problem and has attracted many researchers in both experimental and theoretical investigations on the micro/nano-scale [8–15] using different models, methods and approaches.

At the microscale, the electrostatic interaction force is dominant and has a magnitude that is proportional to the inverse of the square of the gap distance between two electrodes. In [16], Zhang and Zhao used a single-mode and multi-mode analysis method to study the pull-in instability of microstructures under electrostatic loading. The authors showed that the first modal shape is dominant, and the number of modes is not a major factor influencing the computational results of the multi-mode analysis. This observation can be applied to micro/nano-beam systems with a single mode that can provide an estimation with acceptable accuracy for the response compared to multi-mode studies. In fact, this observation is also known in beam systems with large dimensions. The electrostatic field between two electrodes can be affected by the appearance of the fringing field. Hu and Wei [17] considered the fringing effect for calculation of the pull-in voltage of a micro-curved cantilever beam. The analytical results for the pull-in voltage point are obtained based on the energy method. To more accurately evaluate the physical behavior of microbeams, some authors have extended the conventional Euler–Bernoulli beam theory to the Euler–Bernoulli beam model that considers the size scale effect using the modified couple stress theory [18–20]. Rahaeifard *et al* [21] analyzed the static pull-in instability of microcantilevers based on the modified couple stress theory. The effect of the fringing field was also included. The authors' contribution has shown that the use of the modified couple stress theory can reduce the difference between the experimental observations and simulation results based on the classical theory for the static pull-in voltage of the system. Analysis of the nonlinear behavior of electrically actuated MEMS based on the modified couple stress theory can also be found in some works, such as [22–26]. In [27], Rhaeifard and Ahmadian developed the theory of a strain gradient to study the pull-in instability of microcantilevers. The Hamiltonian's principle is utilized to derive the equation of motion of a Euler–Bernoulli microbeam. The model of the beam may also be concerned with functionally graded materials. Jia *et al* [28] considered a poly-SiGe functionally graded microbeam under combined electrostatic intermolecular forces and axial residual stress in a situation that emphasized the effects of the ground electrode shape, position-dependent material composition and geometrically nonlinear deformation of the microbeam. Tahani and Askari [29] introduced a simple and accurate formula to predict the pull-in instability voltage point of a fully clamped

electrostatically actuated nano/micro-beam based on the linear universal graph approach. The nonlinear pull-in instability of the micro/nano systems with surface energy is explored by Wang *et al* [30]. The authors derived the governing equation of motion using the modified couple stress theory and showed that the pull-in voltage and fundamental frequency of the plate are considerably enhanced by the material length scale, surface energy and geometrically nonlinear deformation. The coupling between the surface energy and dispersion forces on nonlinear size-dependent pull-in instability of functionally graded micro/nano-switches was considered by Attia and Mohamed [31]. The influences of the material and geometrical parameters on the nonlinear size-dependent electroelastostatic behavior of functionally graded micro/nano-switches were investigated in detail.

At the nanoscale, researchers are interested in the influence of near-field interactions on the response of systems: typically, the influence of van der Waals interactions on response characteristics, including the pull-in instability phenomenon [32, 33]. The van der Waals force is known to be a weak force of attraction between electrically neutral atoms or molecules. They are much weaker than electrostatic forces. For NEMS structures where the distance between the electrode plates is relatively small, the van der Waals force becomes evident and has a significant influence on the structural behavior. In terms of magnitude, the van der Waals force is inversely proportional to the cube of the distance difference between two electrode plates and depends on the electrode material. The influence of van der Waals forces on the system response has been studied in various NEMS systems, with emphasis on the geometrical and material modeling aspects of the systems that act on the change in pull-in characteristics. Rotkin [34] studied the effect of the van der Waals force on the pull-in gap and obtained analytical expressions for the pull-in gap and voltage of a nano-electromechanical system. The influence of van der Waals forces on the pull-in parameters of cantilever-type nanoscale electrostatic actuators was investigated by Ramezani *et al* [35] using a distributed parameter model associated with the Green's function method. Findings on the clamping instability in a carbon nanotube mechanical resonator under the effect of van der Waals force were published in 2014 by Aykol *et al* [36]. When the temperature was cycled between 340 K and 375 K, the hysteretic clamping and unclamping phenomena of the nanotube device could be observed and led to a result with a discrete drop in the mechanical resonance frequency of the order of 5–20 MHz. Bhojawala and Vakharia [37] explored the effects of van der Waals forces on the pull-in voltage, frequency tuning and frequency stability of NEMS devices using the Euler–Bernoulli beam model and the reduced-order model method. They showed that the presence of van der Waals forces reduces the pull-in voltage and pull-in displacement. The reduction in the pull-in parameters due to van der Waals force is considerable in beams with high axial compressive force. In [38], the pull-in instability of cantilever and fixed-fixed nano-switches subjected to electrostatic and intermolecular forces was investigated. The effect of small length scale was taken into account using a

hybrid nonlocal Euler–Bernoulli beam model. A new model developed by Yazdanpanahi *et al* [39] introduced a balanced liquid layer to the system of electrostatic doubly clamped nano actuators. The governing equation of motion of the actuator was established using the continuous beam model and then using the modified Adomian decomposition method to obtain analytical solutions for the buckling and pull-in instability of the actuator. Moradweysi *et al* [40] applied the modified Adomian decomposition method to analyze the pull-in instability of nano-switches using nonlocal Timoshenko beam theory. Mirkalantari *et al* [41] developed a model based on the strain gradient, by considering surface stress effects, to analyze the pull-in instability behavior of a rectangular nanoplate. The flexoelectric effect is one of the most important phenomena created in micro/nanoscale beams and affects the results of the beam's response. Recently, in [42], the pull-in instability in the beam-type nano-electromechanical systems has been investigated under the flexoelectric effect. That research shows that the pull-in voltage increases due to the flexoelectric effect reduction; and in the indirect state of the flexoelectric effect, the increase in the applied voltage causes a decrease in the pull-in voltage. A contribution to the nonlocal theory for the study of nanostructures with pull-in instability is the two-phase model that can capture the size-dependent behavior of a nanotweezer device and can provide a more accurate description of its electromechanical response [43]. The model has enhanced the understanding of the behavior of devices at the nanoscale and has paved the way for more accurate predictions and design considerations in this field.

With regard to the problems of vibrations of nano/micro-structures, the pull-in instability phenomenon is evaluated via the amplitude and frequency of oscillation. In free vibrations, the frequency evaluation is an important criterion and needs to be dealt with by reliable methods with high accuracy [44]. Unlike the static problem, the pull-in response in the dynamic problems can be more complicated due to the requirement that the maximum point of the response must not exceed the gap distance between the two electrode plates, whereas this maximum point depends on both the amplitude and average response of the system. Here, the response average has the same meaning as the solution in the static problem. Another difference when compared to the static problem is that most dynamic systems can vibrate with different behavior, and they are nonlinear and belong to classes of strongly nonlinear systems; therefore, finding a solution for strongly nonlinear systems is relatively difficult. Note that dynamic problem models can be extensions or generalizations from previously known static models with electrostatic interaction effects, van der Waals interaction effects and Casimir effects, and beam models using modified couple stress theory and nonlocal theory [45, 46]. The free vibration systems without damping often lead to periodic motions; the obtained solution is periodic, and the period can be found approximately [47]. Free vibration of a clamped nanobeam in the presence of the Casimir force was studied by Askari and Tahani [48] using a combination of the Galerkin decomposition procedure and the homotopy analysis method (HAM). The obtained pull-in response

was shown to agree well with those obtained by the fourth-order Runge–Kutta method. Askari *et al* [49] realized that there are no presented criteria for predicting the threshold of the frequency behavior change in micro/nano-beams. They proposed a new frequency criterion for predicting the type of frequency behavior in a double-clamped beam-type N/MEMS under the van der Waals attraction using the HAM approach. For beam-type N/MEMS, Ismail and Cveticanin [50] used the higher-order Hamiltonian approach to solve double-clamped beams subjected to the van der Waals interaction. An approach based on the homotopy perturbation method was introduced by Shamsmohammadi *et al* [51] to study the response of nano/micro-beams under the effects of the van der Waals interaction force.

It can be said that most of the systems in the fields of engineering and technology are nonlinear, especially the nonlinear systems appearing in N/MEMS that belong to the strongly nonlinear type. There have been many efforts proposed and developed to find approximate solutions to such nonlinear systems, such as the perturbation method [52], the harmonic balance method [53], the multiscale method [54], the Adomian decomposition method [55] and the variational iteration method [56]. For nonlinear systems of the strongly nonlinear type, a method that has been observed and has attracted attention in recent years is the elliptic balance method (EBM). The EBM is a generalization of the harmonic balance method (HBM), in which the utilized trial function is the Jacobi elliptic function instead of the usual trigonometric functions [57]. The use of the Jacobi elliptic function leads to the interesting result that it can be used to represent exact solutions of undamped cubic Duffing oscillators [58], and even undamped quadratic nonlinear systems [59], or mixed-type systems containing both the second- and third-order nonlinearities [60]. In [61], Elías–Zúñiga presented exact solutions to a cubic-quintic Duffing oscillator equation based on the use of Jacobi elliptic functions. Recently, for free vibrations, Lenci [62] obtained an exact analytical solution of a system consisting of two coupled Duffing oscillators, which served as a basis for investigating the frequency characteristics with different system parameters. In systems subjected to external excitations, Vakakis and Blanchard [63] obtained an exact steady-state solution for a strongly nonlinear Duffing oscillator with arbitrary viscous damping. A review on the Jacobi elliptic functions and related research results are summarized in [64]. Because of the relatively unique properties of the Jacobi elliptic function, it has enabled the EBM to be applicable to a wide variety of nonlinear systems: both free and forced vibration systems. The primary indications on the solution methods for nonlinear systems based on the EBM can be found in [65]. Simple polynomial-type systems are usually considered to find periodic solutions or limit cycles in the process of calculation implementation for the EBM. In [66], Hsu applied the EBM to find approximate solutions of a Duffing system subjected to periodic excitation. Using the EBM in a case of external excitation, Elías–Zúñiga [67] obtained a general solution of the forced damped Duffing equation by assuming that the modulus of the Jacobian elliptic function is slowly varying as a

function of time. Elías-Zúñiga [68] developed the EBM for two-degrees-of-freedom systems with external excitation in the form of elliptic functions. Furthermore, Elías-Zúñiga *et al* [69] sought approximate frequency solutions using the EBM for a nonlinear singular oscillator. The authors showed a significant improvement in the error of the solution compared to the exact solution of the system. The system explored by Elías-Zúñiga, however, is relatively simple. In [70], Wu and Tang developed a modified EBM and used it to solve the force and displacement transmissibility of a two-stage geometrically nonlinear vibration isolation system. They showed that the developed method can reduce the truncation error in the harmonic balance solution and can improve the accuracy of the resonance regimes of the amplitude–frequency curve and transmissibility. Most recently, the EBM was implemented to obtain the steady-state response of a nonlinear metamaterial unit cell [71] in wave propagation problems. This is a relatively new approach for finding the dispersion relationship of a nonlinear metamaterial with Duffing-type cubic nonlinearity at the attached oscillator.

Previous studies on N/MEMS have provided approximate expressions of the response in time or approximate expressions of the frequency with an accuracy that may not be as expected because of the large nonlinear characteristics of the system. To improve the response accuracy and to be able to accurately predict the pull-in response, finding a suitable method with high accuracy is a reasonable requirement. In this paper, our solution idea is based on the observation that, in a nonlinear and undamped free oscillation system, a Duffing-type nonlinear system can give an exact solution through the representation of generalized trigonometric functions based on Jacobi elliptic functions [64], as presented in the above analysis for the EBM. This idea is natural and is expected to be applicable for finding approximate solutions of strongly nonlinear systems, even systems that do not have the known Duffing form. Therefore, in this study, we have applied the method of elliptic balance based on the elliptic function to find approximate responses in time and frequency of a beam-type N/MEMS subjected to electrostatic and van der Waals interaction forces. Parameter analysis is carried out to assess the accuracy of the proposed method.

The nonlinearity of N/MEMS may come from two main sources. The first comes from the geometrical nonlinearity at which large deformations occur due to the axial stretching effect of the beam. This source often appears in N/MEMS structures with clamped boundary conditions at both ends, and therefore the axial stretching force is significant. For N/MEMS structures with cantilever beams, the nonlinearity due to the axial stretching effect is usually small and can be neglected in many situations. The second nonlinearity source comes from the interaction forces between the two electrodes of the structure, such as the electrostatic interaction, van der Waals interaction and Casimir interaction. The electrostatic interaction is inversely quadratic with the distance between the two electrodes. The van der Waals interaction is inversely cubic; whereas the Casimir interaction is inversely quartic with the

distance between the two electrodes. This leads to the fact that the governing differential equation of vibration is a strongly nonlinear equation with a nonlinear function of the fractional form. Solving such a differential equation is not an easy task, even with the search for approximate solutions. A simple treatment is to approximate the polynomial form based on the Taylor expansion of the above fractional functions, and then use known approximation methods for nonlinear systems of the polynomial form to find the solution of the original system. The disadvantage of this approach is that it is only suitable in the case where the ratio of the beam deflection to the distance between two electrodes is relatively small, and the calculation error will increase when the beam deflection response is large and near the pull-in position. The characteristic of the above solution is that the system has been reduced to a weak nonlinear property, which may be fundamentally different from the original strong nonlinear characteristics. The harmonic balance method is one of the useful tools for finding the frequency of a strongly nonlinear oscillating system, but it is mostly studied for polynomial-type systems or can be reduced to a polynomial form in some contexts. The advantage of this method is that it does not require small parameters like the perturbation methods. However, to improve the accuracy of the harmonic balance method, higher-order harmonic terms need to be taken into account. This leads to the calculation process becoming a complicated problem. The EBM is a new idea for solving the systems encountered in the field of N/MEMS. Due to its generality compared to the traditional harmonic balance method, the EBM is expected to give a more accurate solution from the following observation: it can be seen that the undamped free Duffing vibration oscillator is a conservative system, and the exact representation of this Duffing equation through the elliptic function representation shows that the Jacobian elliptic function is closely related to the energy conservation of the system. This suggests that, in some circumstances, the solution trajectory of a conservative nonlinear system can be approximated through the elliptic function. In terms of the intrinsic representation of the solution in our present approach, the approximate solution is represented similarly to the well-known approximation of the EBM, but it is worth noting that the approximate representation has an additional free term. The free term is emphasized because, in most of the previous studies, it is ignored due to the mathematical assumption that the mean value of the response is zero. A zero-mean value would be unreasonable for the N/MEMS system because the system may oscillate around the non-zero equilibrium point due to the interaction forces, especially when the applied voltage of the system increases. Such a representation leads to the fact that we must solve the equation with an increased number of unknowns. This is different from the representations of the solutions in previous studies of MEMS systems. Due to this representation, we have analyzed and discovered the point with zero amplitude before the pull-in phenomenon occurs during the process of increasing the applied voltage on the system.

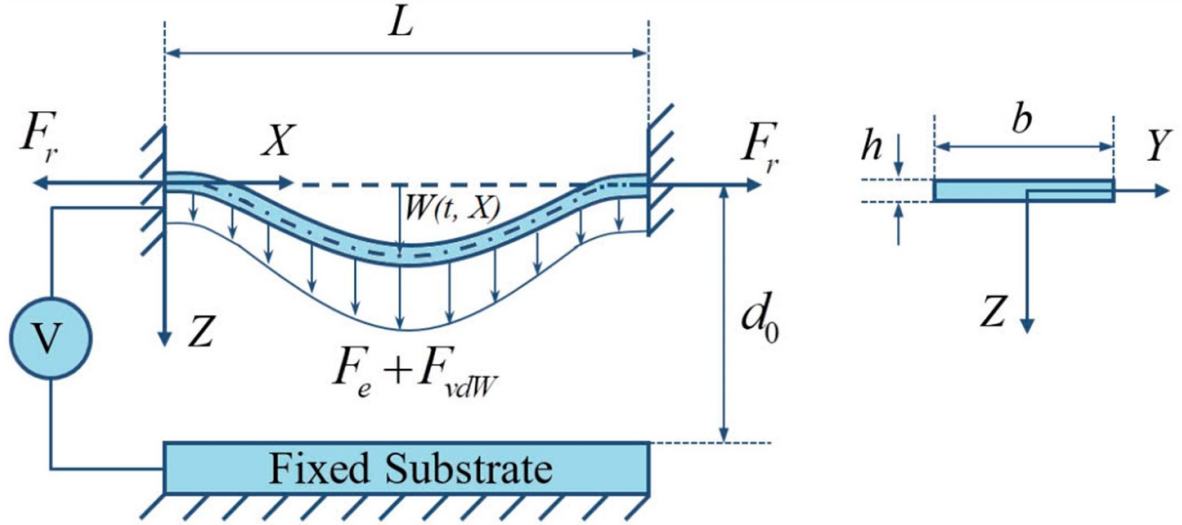


Figure 1. Modeling of a clamped-clamped micro/nano-beam with a rectangular cross section under the effects of electrostatic and van der Waals forces.

2. Governing equation of beam vibration

Consider a clamped-clamped micro/nano-beam interacting with a fixed substrate at distance d_0 by the electrostatic and van der Waals interactions, as shown in figure 1. The beam and fixed substrate play as two electrodes in configurations of structures at the micro/nanoscale, such as sensors and actuators. The first electrode is modeled as a deformable beam for which the large deflection can be present due to the force interactions, whereas the second electrode is assumed to be fixed and the body deformation is not considered. The governing equation for the beam is given by [49]

$$EI \frac{\partial^4 W}{\partial X^4} + \rho b h \frac{\partial^2 W}{\partial t^2} = \left(F_r + \frac{E b h}{2L} \int_0^L \left(\frac{\partial W}{\partial X} \right)^2 dX \right) \frac{\partial^2 W}{\partial X^2} + F_e + F_{vdW} \quad (1)$$

where $W = W(t, X)$ is the deflection of the beam depending on time t and position X along the length L of the beam; E is the Young's modulus; I is the second-order inertia moment of the rectangular cross section with dimension $b \times h$; and ρ is the mass density of the beam material. The electrostatic force per unit length of the beam is denoted as F_e and is expressed as a function of the distance difference $d_0 - W$ between the beam and fixed substrate as follows

$$F_e = \frac{\varepsilon b V^2}{2(d_0 - W)^2} \quad (2)$$

where $\varepsilon = 8.854 \times 10^{-22}$ (F/m) is the dielectric constant of the medium, and V is a polarized DC applied voltage. The van der Waals force F_{vdW} is given by [49]

$$F_{vdW} = \frac{A_H b}{6\pi (d_0 - W)^3} \quad (3)$$

where A_H is the Hamaker constant [72]. The value of the Hamaker constant can be taken in the range $(30 - 50) \times 10^{-20}$ J for two identical metal media (Ag, Au, Cu) interacting cross vacuum (air). The axial residual force is denoted as F_r and is given by:

$$F_r = \sigma_r b h \quad (4)$$

where σ_r expresses the axial residual stress of the beam. The second integral expression on the right-hand side of equation (1) represents the geometrical nonlinearity related to the axial stretching effect of the mid-plane of the beam. Let us introduce dimensionless quantities and denote as

$$x = \frac{X}{L}, w = \frac{W}{d_0}, \tau = \frac{t}{T}, N = \frac{12 F_r L^2}{E b h^3}, \alpha = 6 \left(\frac{d_0}{h} \right)^2, \beta = \frac{6 \varepsilon V^2 L^4}{E d_0^3 h^3}, \lambda = \frac{2 A_H L^4}{\pi E h^3 d_0^4}, T = \sqrt{\frac{\rho b h L^4}{EI}}. \quad (5)$$

The dimensionless quantity N expresses the axial residual force; three quantities, α , β , λ , represent the gap distance between the beam and substrate, electrostatic and van der Waals force parameters, respectively. Here, T is the vibrational period corresponding to the fundamental frequency of the beam, i.e. $T = 2\pi / \omega_0$ with $\omega_0 = 2\pi \sqrt{EI / (\rho b h L^4)}$. The governing equation (1) is transformed in the following dimensionless form:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial \tau^2} = \left(N + \alpha \int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 dx \right) \frac{\partial^2 w}{\partial x^2} + \frac{\beta}{(1 - w)^2} + \frac{\lambda}{(1 - w)^3}. \quad (6)$$

In the following, we use the Galerkin method [73–75] with a single mode to transform equation (6) to the corresponding

form of a single-degree-of-freedom system. Accordingly, the dimensionless deflection $w(\tau, x)$ is expressed as a product of the modal function $\psi(x)$ and a time-varying function $u(\tau)$:

$$w(\tau, x) = \psi(x) u(\tau). \quad (7)$$

Note that in the general form, the deflection $w(\tau, x)$ can be expressed as a linear combination of the modal functions ψ_i and time-varying functions $u_i(\tau)$. In such a situation, the calculation process is similar to the single-mode case. The modal function $\psi(x)$ is found from the conditions of free vibration of the clamped-clamped beam:

$$\psi(x) = \frac{1}{\varphi_1(0.5)} \varphi_1(x) \quad (8)$$

where the function $\varphi_1(x)$ is given by:

$$\begin{aligned} \varphi_1(x) = & \cosh(\sqrt{\Omega_1}x) - \cos(\sqrt{\Omega_1}x) \\ & - \frac{\sinh(\sqrt{\Omega_1}) + \sin(\sqrt{\Omega_1})}{\cosh(\sqrt{\Omega_1}) - \cos(\sqrt{\Omega_1})} \left(\sinh(\sqrt{\Omega_1}x) - \sin(\sqrt{\Omega_1}x) \right). \end{aligned} \quad (9)$$

Here, the parameter Ω_1 takes a value in which $\sqrt{\Omega_1} = 4.73$ for the clamped-clamped beam in the first modal form of free vibration. The second-, third-, fourth- and fifth-order forms of free vibration are $\sqrt{\Omega_1} = 7.8532, 10.9956, 14.1372, 17.2788$, respectively. The form in equation (7) is selected because it yields an observation that the function $u(\tau)$ is expressed for the deflection of the mid-point of the beam, i.e. $w(\tau, 0.5) = \psi(0.5)u(\tau) = 1 \times u(\tau) = u(\tau)$.

By substituting equation (7) into equation (6), multiplying both sides of the resulting equation by $\psi(1-w)^3$ and noting the property $\psi^{(4)}(x) = \omega_0^2 \psi(x)$, we arrive at

$$\begin{aligned} & (\psi^2 - 3\psi^3 u + 3\psi^4 u^2 - \psi^5 u^3) \ddot{u} + (\beta \psi^2 + \psi \psi^{(4)} - N \psi \psi'') u \\ & + (3\psi^{(4)} \psi^2 + 3N \psi^2 \psi'') u^2 + ((3\psi^{(4)} \psi^3 - N \psi^3 \psi'') \\ & - \alpha p \psi \psi'') u^3 + (-\psi^{(4)} \psi^4 + N \psi^4 \psi'' + 3\alpha p \psi^2 \psi'') u^4 \\ & - 3\alpha p \psi^3 \psi'' u^5 + \alpha p \psi^4 \psi'' u^6 - (\beta + \lambda) \psi = 0 \end{aligned} \quad (10)$$

where we denote p as follows for simplicity

$$p = \int_0^1 (\psi')^2 dx. \quad (11)$$

Taking integration on both sides of equation (10) in the interval $[0, 1]$, we obtain the following equation of the time-varying function $u(\tau)$:

$$\begin{aligned} & (a_0 + a_1 u + a_2 u^2 + a_3 u^3) \ddot{u} + b_0 + b_1 u + b_2 u^2 + b_3 u^3 + b_4 u^4 \\ & + b_5 u^5 + b_6 u^6 = 0 \end{aligned} \quad (12)$$

with initial conditions for $u(\tau)$:

$$u(0) = A, \dot{u}(0) = 0 \quad (13)$$

and the coefficients a_i, b_i given by:

$$\begin{aligned} a_0 = & \int_0^1 \psi^2 dx, \quad a_1 = -3 \int_0^1 \psi^3 dx, \quad a_2 = 3 \int_0^1 \psi^4 dx, \quad a_3 = - \int_0^1 \psi^5 dx \\ b_0 = & -(\beta + \lambda) \int_0^1 \psi dx \\ b_1 = & \beta \int_0^1 \psi^2 dx + \int_0^1 \psi^{(4)} \psi dx - N \int_0^1 \psi'' \psi dx \\ b_2 = & -3 \int_0^1 \psi^{(4)} \psi^2 dx + 3N \int_0^1 \psi'' \psi^2 dx \\ b_3 = & 3 \int_0^1 \psi^{(4)} \psi^3 dx - 3N \int_0^1 \psi'' \psi^3 dx - \alpha \int_0^1 \psi'^2 dx \int_0^1 \psi'' \psi dx \\ b_4 = & - \int_0^1 \psi^{(4)} \psi^4 dx + N \int_0^1 \psi'' \psi^4 dx + 3\alpha \int_0^1 \psi'^2 dx \int_0^1 \psi'' \psi^2 dx \\ b_5 = & -3\alpha \int_0^1 \psi'^2 dx \int_0^1 \psi'' \psi^3 dx \\ b_6 = & \alpha \int_0^1 \psi'^2 dx \int_0^1 \psi'' \psi^4 dx. \end{aligned} \quad (14)$$

It is seen that the integrals for the coefficients a_i, b_i can be calculated using an appropriate numerical method, for example, the Simpson's method with a reliable accuracy [76, 77]. The coefficients a_i ($i = 0, 1, 2, 3$) only depend on the shape function $\psi(x)$, whereas the coefficients b_i ($i = 0-6$) depend on both systems' parameters and the modal function. Therefore, the effect of system parameters will be reflected via the coefficients b_i ($i = 0-6$), in which they constitute components of the nonlinear equation (12). Equation (12) takes the form of a conservative system in which the damping effect is absent. The conservation law yields the fact that the system response $u(\tau)$ has a form of periodic motion with certain frequencies. In the following, we find the vibrational frequency of the system, equation (12), using the EBM.

3. Elliptic balance and solution method

Equation (12) is transformed into the following form with frequency ω :

$$\begin{aligned} & \omega^2 (a_0 + a_1 u + a_2 u^2 + a_3 u^3) u'' + b_0 + b_1 u + b_2 u^2 + b_3 u^3 \\ & + b_4 u^4 + b_5 u^5 + b_6 u^6 = 0 \end{aligned} \quad (15)$$

where the double prime denotes the derivative of response u corresponding to time τ .

Assume that the solution of response $u(\tau)$ takes the following form

$$u(\tau) = D_0 + D_1 \text{cn}(\tau, k^2) \quad (16)$$

where cn is an elliptic cosine function with two arguments τ and k [64]. The quantity k^2 is called the modulus of the function. Two

coefficients, D_0 , D_1 , and the modulus k^2 are quantities that must be determined. The free term D_0 is added to the expression of $u(\tau)$ instead of only the elliptic cosine term because the response may vibrate around a non-zero position.

The initial condition, equation (13), gives an equation for D_0 , D_1 , i.e. $D_0 + D_1 = A$. The expressions for the first and second derivatives of $u(\tau)$ according to dimensionless time τ are as follows

$$\begin{aligned} u'(\tau) &= -D_1 \operatorname{sn}(\tau, k^2) \operatorname{dn}(\tau, k^2) \\ u''(\tau) &= -D_1 \left((1 - 2k^2) \operatorname{cn}(\tau, k^2) + 2k^2 \operatorname{cn}^3(\tau, k^2) \right) \end{aligned} \quad (17)$$

where sn , dn are sine and delta Jacobian elliptic functions of two arguments τ , k . Let us denote φ_a as an amplitude function in elliptic trigonometry that satisfies $\cos \varphi_a = \operatorname{cn}(\tau, k^2)$. Using the notation φ_a for calculating elliptic functions of u^i ($i = 1, 2, \dots, 6$), we obtain a trigonometric expression for the left-hand side of equation (15):

$$\begin{aligned} &\omega^2 (a_0 + a_1 u + a_2 u^2 + a_3 u^3) u'' + b_0 + b_1 u + b_2 u^2 + b_3 u^3 \\ &\quad + b_4 u^4 + b_5 u^5 + b_6 u^6 \\ &= h_0 + h_1 \operatorname{cn}(t, k^2) + h_2 \operatorname{cn}^2(t, k^2) + h_3 \operatorname{cn}^3(t, k^2) \\ &\quad + h_4 \operatorname{cn}^4(t, k^2) + h_5 \operatorname{cn}^5(t, k^2) + h_6 \operatorname{cn}^6(t, k^2) \\ &= h_0 + h_1 \cos \varphi_a + h_2 \cos^2 \varphi_a + h_3 \cos^3 \varphi_a + h_4 \cos^4 \varphi_a \\ &\quad + h_5 \cos^5 \varphi_a + h_6 \cos^6 \varphi_a \end{aligned} \quad (18)$$

where h_i ($i = 0, 1, 2, \dots, 6$) are given by

$$\begin{aligned} h_0 &= b_0 + b_1 D_0 + b_2 D_0^2 + b_3 D_0^3 + b_4 D_0^4 + b_5 D_0^5 + b_6 D_0^6 \\ h_1 &= D_1 \left[(b_1 + 2b_2 D_0 + 3b_3 D_0^2 + 4b_4 D_0^3 + 5b_5 D_0^4 + 6b_6 D_0^5) \right. \\ &\quad \left. - \omega^2 D_1 (a_0 + a_1 D_0 + a_2 D_0^2 + a_3 D_0^3) \right. \\ &\quad \left. + 2\omega^2 k^2 D_1 (a_0 + a_1 D_0 + a_2 D_0^2 + a_3 D_0^3) \right] \\ &= h_{10} - \omega^2 h_{11} + 2\omega^2 k^2 h_{12} \\ h_2 &= h_{20} - \omega^2 h_{21} + 2\omega^2 k^2 h_{22} \\ h_3 &= h_{30} - \omega^2 h_{31} + 2\omega^2 k^2 h_{32} \\ h_4 &= h_{40} - \omega^2 h_{41} + 2\omega^2 k^2 h_{42} \\ h_5 &= h_{50} - 2\omega^2 k^2 h_{52} \\ h_6 &= h_{60} - 2\omega^2 k^2 h_{62} \end{aligned} \quad (19)$$

with h_{ij} being

$$\begin{aligned} h_{10} &= D_1 (b_1 + 2b_2 D_0 + 3b_3 D_0^2 + 4b_4 D_0^3 + 5b_5 D_0^4 + 6b_6 D_0^5) \\ h_{11} &= D_1 (a_0 + a_1 D_0 + a_2 D_0^2 + a_3 D_0^3) \\ h_{12} &= D_1 (a_0 + a_1 D_0 + a_2 D_0^2 + a_3 D_0^3) \\ h_{20} &= D_1^2 (b_2 + 3b_3 D_0 + 6b_4 D_0^2 + 10b_5 D_0^3 + 15b_6 D_0^4) \\ h_{21} &= D_1^2 (a_1 + 2a_2 D_0 + 3a_3 D_0^2) \\ h_{22} &= D_1^2 (a_1 + 2a_2 D_0 + 3a_3 D_0^2) \end{aligned}$$

$$\begin{aligned} h_{30} &= D_1^3 (b_3 + 4b_4 D_0 + 10b_5 D_0^2 + 20b_6 D_0^3) \\ h_{31} &= D_1^3 (a_2 + 3a_3 D_0) \\ h_{32} &= D_1^3 (a_2 + 3a_3 D_0) - D_1 (a_0 + a_1 D_0 + a_2 D_0^2 + a_3 D_0^3) \\ h_{40} &= D_1^4 (b_4 + 5b_5 D_0 + 15b_6 D_0^2) \\ h_{41} &= D_1^4 a_3 \\ h_{42} &= D_1^4 a_3 - D_1^2 (a_1 + 2a_2 D_0 + 3a_3 D_0^2) \\ h_{50} &= D_1^5 (b_5 + 6b_6 D_0) \\ h_{52} &= D_1^3 (a_2 + 3a_3 D_0) \\ h_{60} &= D_1^5 b_6 \\ h_{62} &= D_1^4 a_3. \end{aligned} \quad (20)$$

we have identical expressions for $\cos^2 \varphi_a$, $\cos^3 \varphi_a$, $\cos^4 \varphi_a$, $\cos^5 \varphi_a$, $\cos^6 \varphi_a$

$$\begin{aligned} \cos^2 \varphi_a &= \frac{1}{2} + \frac{1}{2} \cos 2\varphi_a \\ \cos^3 \varphi_a &= \frac{3}{4} \cos \varphi_a + \frac{1}{4} \cos 3\varphi_a \\ \cos^4 \varphi_a &= \frac{3}{8} + \frac{1}{2} \cos 2\varphi_a + \frac{1}{8} \cos 4\varphi_a \\ \cos^5 \varphi_a &= \frac{5}{8} \cos \varphi_a + \frac{5}{16} \cos 3\varphi_a + \frac{1}{16} \cos 5\varphi_a \\ \cos^6 \varphi_a &= \frac{5}{16} + \frac{15}{32} \cos 2\varphi_a + \frac{3}{16} \cos 4\varphi_a + \frac{1}{32} \cos 6\varphi_a. \end{aligned} \quad (21)$$

From equation (15) and expressions (18) and (21), creating a harmonic balance for coefficients of $\{1, \cos \varphi_a, \cos 3\varphi_a\}$, we obtain an algebraic system of three equations associated with the fourth equation related to the initial conditions equation, i.e. $D_0 + D_1 = A$, for the unknowns ω , k , D_0 , D_1 as follows:

$$D_0 + D_1 - A = 0 \quad (22.1)$$

$$\begin{aligned} h_0(\omega, k, D_0, D_1) + \frac{1}{2} h_2(\omega, k, D_0, D_1) + \frac{3}{8} h_4(\omega, k, D_0, D_1) \\ + \frac{5}{16} h_6(\omega, k, D_0, D_1) = 0 \end{aligned} \quad (22.2)$$

$$h_1(\omega, k, D_0, D_1) + \frac{3}{4} h_3(\omega, k, D_0, D_1) + \frac{5}{8} h_5(\omega, k, D_0, D_1) = 0 \quad (22.3)$$

$$\frac{1}{4} h_3(\omega, k, D_0, D_1) + \frac{5}{16} h_5(\omega, k, D_0, D_1) = 0 \quad (22.4)$$

where the functions $h_i(\omega, k, D_0, D_1)$ are determined from equation (19). We will solve the system in equation (22) using an appropriate iterative method. Accordingly, we will carry out several algebraic calculation steps. Using the coefficients h_{ij} in equation (20), equations (22.3) and (22.4) are transformed into the following form:

$$\begin{aligned} \lambda_{11} \omega^2 - 2\omega^2 k^2 \lambda_{12} &= \lambda_{13} \\ \lambda_{31} \omega^2 - 2\omega^2 k^2 \lambda_{32} &= \lambda_{33} \end{aligned} \quad (23)$$

where

$$\begin{aligned}
 \lambda_{11} &= h_{11} + \frac{3}{4}h_{31} \\
 \lambda_{12} &= h_{12} + \frac{3}{4}h_{32} - \frac{5}{8}h_{52} \\
 \lambda_{13} &= h_0 + \frac{3}{4}h_{30} + \frac{5}{8}h_{50} \\
 \lambda_{31} &= \frac{1}{4}h_{31} \\
 \lambda_{32} &= \frac{1}{4}h_{32} - \frac{5}{16}h_{52} \\
 \lambda_{33} &= \frac{1}{4}h_{30} + \frac{5}{16}h_{50}.
 \end{aligned} \quad (24)$$

Solving equation (23) for ω^2 , k^2 arrives at

$$\begin{aligned}
 \omega^2 &= \frac{\lambda_{12}\lambda_{33} - \lambda_{13}\lambda_{32}}{\lambda_{12}\lambda_{31} - \lambda_{11}\lambda_{32}} := G_1(D_0, D_1) \\
 k^2 &= \frac{1}{2} \frac{\lambda_{11}\lambda_{33} - \lambda_{13}\lambda_{31}}{\lambda_{12}\lambda_{33} - \lambda_{13}\lambda_{32}} := G_2(D_0, D_1).
 \end{aligned} \quad (25)$$

Because λ_{ij} are functions of only two arguments, D_0 , D_1 , we obtain two analytical expressions of ω^2 , k^2 , depending on the D_0 , D_1 . This result reveals a real meaning that the complexity of system (23) is reduced significantly for our solution calculation. To further simplify system (22), substituting the result in equation (25) for ω^2 , k^2 into equation (22.2) yields

$$\lambda_{01}G_1(D_0, D_1) - 2G_1(D_0, D_1)G_2(D_0, D_1)\lambda_{02} - \lambda_{03} = 0 \quad (26)$$

where

$$\begin{aligned}
 \lambda_{01} &= \frac{1}{2}h_{21} + \frac{3}{8}h_{41} \\
 \lambda_{02} &= \frac{1}{2}h_{22} + \frac{3}{8}h_{42} - \frac{5}{16}h_{62} \\
 \lambda_{03} &= h_0 + \frac{1}{2}h_{20} + \frac{3}{8}h_{40} + \frac{5}{16}h_{60}.
 \end{aligned} \quad (27)$$

Substitution of $D_0 = A - D_1$ from equation (22.1) into equation (26) leads to the following equation of only a single unknown D_1 :

$$\begin{aligned}
 H(D_1) &:= \lambda_{01}G_1(A - D_1, D_1) - 2G_1(A - D_1, D_1) \\
 &\quad G_2(A - D_1, D_1)\lambda_{02} - \lambda_{03} = 0
 \end{aligned} \quad (28)$$

where G_1 , G_2 are given by equation (25). In this study, we select the Newton–Raphson method as an iterative method for finding the unknown D_1 of equation (28) and then use equations (22.1) and (25) to determine the unknowns D_0 , ω , k .

4. Numerical results and discussion

In this section, two situations of the problem of the vibration of N/MEMS are considered. The first is carried out for a microbeam with a clamped-clamped boundary condition. The second is presented for a clamped-clamped nanobeam in which, at the nanoscale, the gap between two electrodes is significantly smaller in comparison with that in the microbeam. The small gap at the nanoscale will lead to the presence of van der Waals interaction forces between the two electrodes.

Table 1. The geometrical and material parameters for the microbeam [49].

L (μm)	b (μm)	h (μm)	d_0 (μm)	E (GPa)	ν	ρ (kg m^{-3})
210	100	1.5	1.18	151	0.3	2332

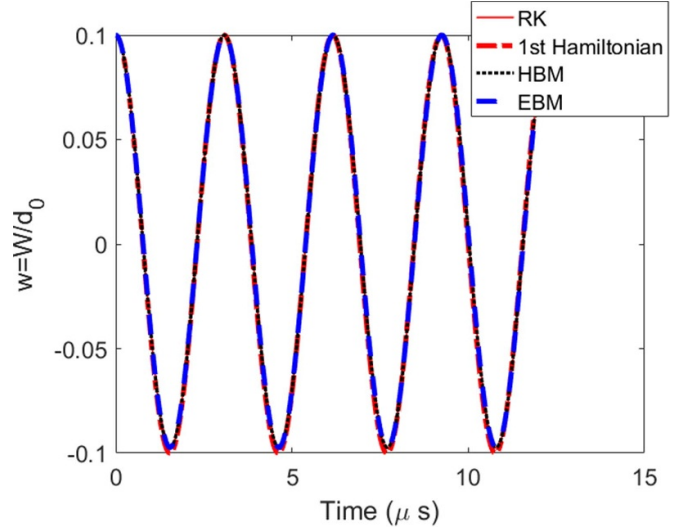


Figure 2. Graphs of the evolution of the time of the response $u(t)$ at the applied voltage $V = 2$ V for the beam length $L = 210$ μm for four different methods.

4.1. Evolutions in time of microbeam deflection

Because the system is conservative, the solution trajectory is periodic in time. We carry out calculations to find the response of the microbeam using four methods: the Runge–Kutta algorithm [78], the first-order Hamiltonian approach [50], the harmonic balance and EBMs [64]. The fourth-order Runge–Kutta algorithm is a common and accurate method to use as a reference for evaluating other methods. The Hamiltonian approach, with a higher-order version for solving the problem of the double-clamped microbeam, was recently developed by Ismail and Cveticanin [50]. However, the explicit form approximation of this approach is only available for the first-order case. We use this result for comparison with our obtained results using the EBM. Our approach with the EBM is developed to find a highly accurate solution for the first time in the vibration problem of micro/nano-beams. The EBM is a generalization of harmonic balance based on the elliptic functions instead of conventional trigonometric functions. Therefore, the result obtained from the harmonic balance method is a special case with the modulus parameter being set to be zero. The system parameters for the microbeam are given in table 1. It is noted that the ratio of the beam width to the height is larger than 5; the Young's modulus will be taken as an effective value, i.e. $E_{\text{eff}} = E/(1 - \nu^2)$. The axial residual force is kept at $F_r = 0.0009$ N, i.e. $\sigma_r = 6$ MPa. The results of the response of the deflection ratio $w = W/d_0$ at the mid-point of the beam are displayed for the four methods in figure 2 when the voltage is taken to be 2 V. The dimensionless time in equations (12) and (16) is converted to the original time in microseconds (μs). To solve equation (12) using the Runge–Kutta algorithm, the initial

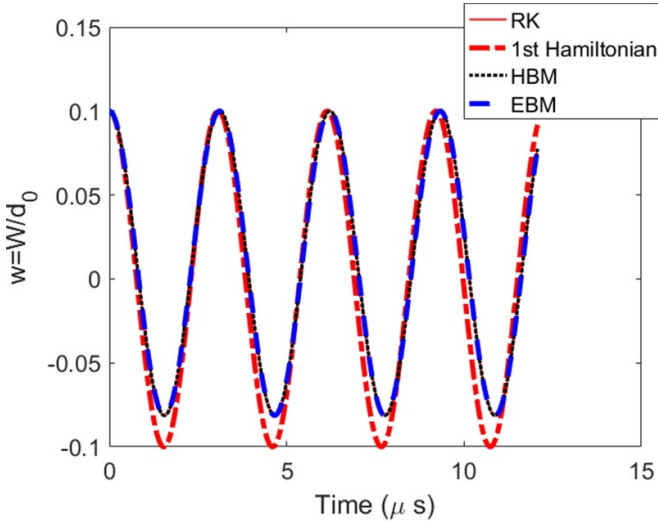


Figure 3. Graphs of the evolution of the time of the response $u(t)$ at the applied voltage $V = 6$ V for the beam length $L = 210$ μm for four different methods.

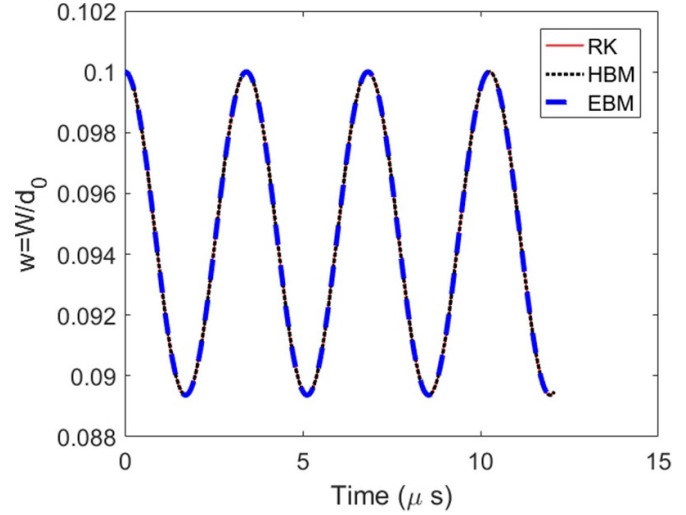


Figure 4. Graphs of the evolution of the time of the response $u(t)$ at the applied voltage $V = 18$ V for the beam length $L = 210$ μm for only three methods: the Runge–Kutta algorithm, the HBM and the EBM.

conditions are set as equation (13) with $A = 0.1$. The periodic solutions show that the response results of the four methods nearly coincide. The voltage value of 2 V applied to the beam of the length $L = 210$ μm in the situation under consideration generates a free vibration with the response's average value in time at nearly zero. This leads to the fact that the approximate representation, equation (16), has the free term D_0 that can be negligible. In the Hamiltonian approach by Ismail and Cveticanin [50], for the first-order approximation, only the periodic term has been retained and no free term is present. In fact, those authors have considered an assumption that the average in the time of the response is zero. If not, an added step with translation transformation for the response is performed, in which the system is converted to the conventional version with zero mean in time.

The results in figure 3 are obtained for an applied voltage at 6 V. If using the first-order form as presented by Ismail and Cveticanin [50], in which the non-zero mean is absent, a large error will occur in the solution response. With the required approximation, equation (16), however, the EBM provides high accuracy with the reference results obtained from the Runge–Kutta method. It is seen that the harmonic balance method also exhibits relatively close tracking to the Runge–Kutta solution.

If we increase the applied voltage to the system, the balance point will be pushed further away due to the appearance of stronger electrostatic force interactions, i.e. the D_0 value increases. Figure 4 presents the solution responses for the above methods, except for the Hamiltonian approach, because its deviation of the first approximation without the free term is quite large. The situation for the Hamiltonian approach will be improved if the free term is considered.

4.2. Comparison of frequency

Figure 5 demonstrates that the frequency curve depends on the applied voltage V with different values of the beam length parameter L with the Runge–Kutta and EBM solutions. The

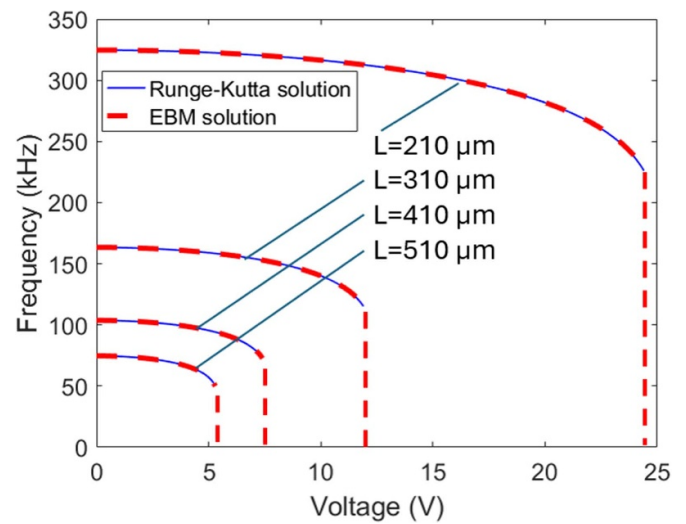


Figure 5. Frequency curves (in kHz) versus the applied voltage V (in V) with different microbeam lengths. The dashed lines show the frequency obtained from the EBM approach; the thin solid lines represent the Runge–Kutta solution. The straight dashed line segments intersecting the horizontal axis show the pull-in instability positions of the frequency curve.

dashed lines depict the pull-in behavior of the vibrational frequency in kHz using the EBM approach. Firstly, the frequency decreases with the increase in the voltage V . If the applied voltage V tends to a threshold value, called the pull-in instability voltage, V_{PI} , the frequency will drop suddenly to the zero point. The values of V_{PI} for four cases of $L = 210, 310, 410$ and 510 μm are found to be 24.4, 11.9, 7.4 and 5.3 V, respectively. To evaluate the accuracy of the EBM approach, the Runge–Kutta solutions are also displayed in figure 5. The two methods are nearly identical. The decrease in frequency as the applied voltage increases reflects a softening effect of the beam vibration, leading to an increase in the vibrational period of the microbeam. This reveals the fact

Table 2. Frequency results of the microbeams obtained from the following methods: (1) the Runge–Kutta solution; (2) the experimental data; (3) the COMSOL simulation solution; (4) the HAM; (5) the Hamiltonian approach; (6) the harmonic balance method; (7) the present EBM. The relative errors of the HAM, Hamiltonian, HBM and EBM are also calculated for four cases 1, 2, 3 and 4, corresponding to different lengths of the beam.

Case	L (μm)	V (V)	Runge–Kutta (kHz)	Experiment [12, 49] (kHz)	COMSOL solution (kHz) [49]	HAM [49] (kHz)	Hamiltonian approach [50] (kHz)	HBM (kHz)	Present EBM (kHz)
1	210	6	321.7741	322.05	321.565	321.565 (0.0650%)	325.9095 (1.2852%)	321.8172 (0.0134%)	321.8035 (0.0091%)
2	310	3	161.8610	163.22	161.833	161.833 (0.0173%)	164.0949 (1.3801%)	161.9105 (0.0306%)	161.9036 (0.0263%)
3	410	2	102.6038	102.17	101.204	101.204 (1.3643%)	104.2668 (1.6208%)	102.6476 (0.0427%)	102.6434 (0.0386%)
4	510	2	73.0461	73.79	73.111	73.111 (0.0888%)	75.3673 (3.1777%)	73.0802 (0.0467%)	73.0782 (0.0439%)

that under the applied voltage at which the pull-in instability phenomenon has not yet occurred, the beam becomes more susceptible to lower frequencies in a range of limited frequency. This frequency range will extend to higher frequencies with smaller beam length values. The point at which the pull-in instability occurs, called the pull-in frequency, is denoted as f_{PI} . The values of f_{PI} for $L = 210, 310, 410$ and $510 \mu\text{m}$ are 227.1, 115.6, 72.71 and 49.98 kHz, respectively.

For further information on the accuracy of the present method based on the EBM approach, several numerical and experimental results are presented here. The experimental data and COMSOL simulation solution are taken from [12, 49], as reported by Askari *et al.* In [49], the authors applied the HAM to find approximate solutions of microbeam vibration. Their results are also presented in the seventh column of table 2 for our reference purposes. The error percentages for the HAM, Hamiltonian, HBM and EBM approaches are calculated as follows:

$$\text{err} (\%) = \frac{|f_{RK} - f_{\text{approx}}|}{f_{RK}} \times 100\% \quad (29)$$

where f_{RK} is the Runge–Kutta solution; and f_{approx} denotes the approximate solutions obtained from the above approximate analytical methods. It is noted that in table 2, the vibrational frequency of the microbeam using the EBM is represented by f_{EBM} in kHz instead of the frequency ω_{EBM} . The relationship between f_{EBM} and ω_{EBM} is given by:

$$f_{EBM} = \frac{\omega_{EBM}}{4 \times \text{ellipke}(k^2)} \quad (30)$$

where $\text{ellipke}(k^2)$ is a complete elliptic integral of the first kind with the modulus k^2 :

$$\text{ellipke}(k^2) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}. \quad (31)$$

In table 2, for the first case with beam length $L = 210 \mu\text{m}$ and voltage $V = 6$ V, the relative error of the frequency obtained from the EBM is the smallest compared to the three methods: the HAM, the Hamiltonian approach and the HBM. The accuracy of the HAM depends on the order of approximation. The frequency

value from the HAM [49], calculated for a five-term approximation, yields an error of 0.065%. This error is considered to be relatively small, and it exhibits the acceptable accuracy of the HAM. The frequency value obtained from the Hamiltonian approach is slightly higher than that of the HAM, at the level of 1.2852%. In the representation of the Hamiltonian approach, the first-order approximation in the form of a harmonic function is utilized for computation purposes in table 2. In the Hamiltonian approach, the accuracy can be further improved if higher-order approximations are used. The harmonic balance method for $L = 210 \mu\text{m}$ gives an error of 0.0134%, which is considered to be relatively small, but still larger than that of the EBM. It can be seen that with a short beam length of $L = 210 \mu\text{m}$, a relatively high voltage is needed for the occurrence of pull-in instability. Here, the pull-in voltage is found to be 24.4 V: a relatively high voltage compared to the voltage value $V = 6$ V in the experimental results [12, 49]. At $V = 6$ V, the amplitude of the beam vibration may still be small compared to the amplitude of the vibration near the pull-in voltage. As the beam length increases, for example for $L = 510 \mu\text{m}$ at $V = 2$ V, the vibrational frequency decreases. The frequency obtained from the EBM is 73.0782 Hz, with a relatively small error compared to the frequency received from the Runge–Kutta method, i.e. 0.0439%. The relative errors obtained from the HAM, Hamiltonian approach and HBM are 0.0888%, 3.1777% and 0.0467%, respectively. All have larger relative errors than the EBM in this case. However, it should be noted that the errors of the HAM, HBM and EBM are almost equivalent; only the error from the Hamiltonian approach is larger. The experimental and COMSOL simulation results give relatively close frequency values and are close to the results obtained from the Runge–Kutta method. If the Runge–Kutta result is taken as a reference, the relative differences between the COMSOL and the Runge–Kutta solutions for beam lengths $L = 210, 310, 410$ and $510 \mu\text{m}$ are 0.0650%, 0.0173%, 1.3643% and 0.0888%, respectively; whereas the relative differences between the experimental and the Runge–Kutta results are 0.0857%, 0.8396%, 0.4228% and 1.0184%, respectively. Overall, the above-mentioned methods demonstrate acceptable accuracy. Although, the Hamiltonian approach has a high accuracy at the level of 3% that is acceptable in engineering applications, we can see that the results obtained from the EBM approach have even higher accuracy, below 0.1%;

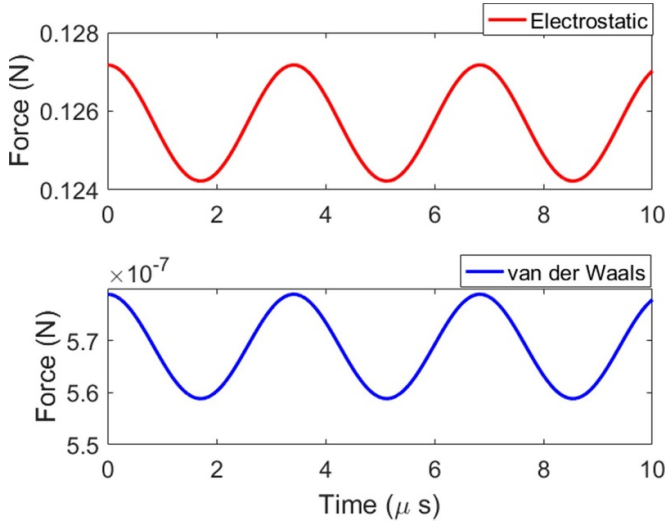


Figure 6. Comparisons of the magnitude values between the electrostatic and van der Waals interaction forces in a microbeam for length $L = 210 \mu\text{m}$ and voltage $V = 0.75 V_{\text{PI}} = 18 \text{ V}$ (i.e. $V_{\text{PI}} = 24.4 \text{ V}$).

in particular, for $L = 210 \mu\text{m}$, the obtained error of the EBM is superior to the other methods, i.e. 0.0091%. This reveals a demonstration that the EBM is a reliable method for studying the vibration of nonlinear systems within the framework of the vibrating microbeam problem under consideration.

The numerical solution shows that in the microbeam system with a large gap width, the effect of van der Waals forces is not considerable and can be negligible. Therefore, the contribution of electrostatic forces plays an important role in the response of beam under the applied voltage. For illustration, we carry out numerical calculations for the length $L = 210 \mu\text{m}$ and the voltage is kept at $0.75 V_{\text{PI}} = 0.75 \times 24.4 = 18 \text{ (V)}$. The results of electrostatic and van der Waals forces are plotted in figure 6. The graphs of both electrostatic and van der Waals forces exhibit the periodic changes in time. For the magnitude, however, the electrostatic force plays a dominant role and these force values are much larger than the van der Waals forces, for example, the maximum value of the electrostatic force is 0.1272 N whereas that of the van der Waals force is $0.5788 \times 10^{-6} \text{ N}$, a very small value in comparison with the value 0.1272 V. It is observed that, at near pull-in instability voltage, at 75% V_{PI} , the electrostatic force can reach 0.1272 N and can be considered as a relatively large value in the situation where the axial residual force is taken to be 0.0009 N.

4.3. The approximate solution of the EBM

In the approximation form, equation (16), of the system response, three unknowns, D_0 , D_1 and k^2 need to be identified. The unknown D_0 has a non-zero value and reflects the mean deflection of the vibrating microbeam. In the static interaction problem in which the inertia term $\rho b h (\partial^2 W / \partial t^2)$ is not present in the governing equation (1), the unknown D_0 represents the static deflection of the beam. With regard to the physical meaning,

Table 3. The change in the coefficients D_0 , D_1 with four values of applied voltage for the beam $L = 210 \mu\text{m}$, and initial conditions $A = 0.1$ for numerical calculations.

Voltage V (V)	D_1	D_0
2.0	0.0986	0.0014
6.0	0.0907	0.0093
18.0	0.0053	0.0947
24.4	-0.1418	0.0218

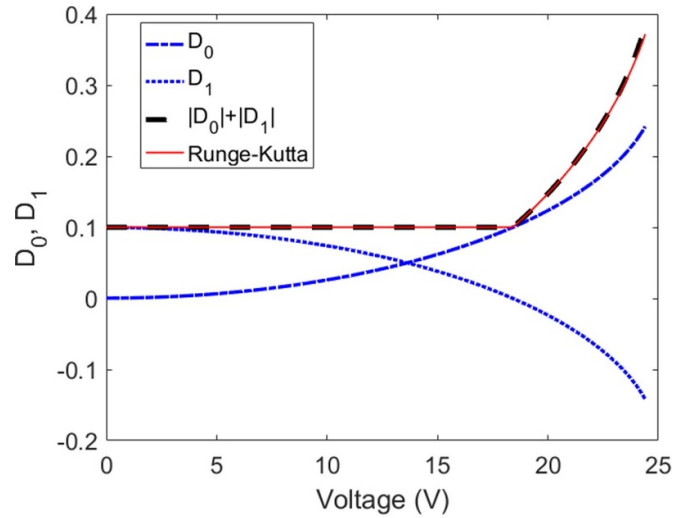


Figure 7. Illustration of three curves, including the free term D_0 , the coefficient D_1 and the maximum point of the response $u(\tau)$ for the beam length $L = 210 \mu\text{m}$.

the value of D_0 must be positive. The quantity $|D_1|$ of the solution $u(\tau) = D_0 + D_1 \text{cn}(\tau, k^2)$ plays a role as an amplitude of the function $\text{cn}(\tau, k^2)$. The value of D_1 may be positive or negative. This is because the initial condition $A = u(0) = D_0 + D_1$, gives $D_0 = A - D_1$. The negative sign of D_1 shows that the mean of the response is pushed on the higher level in comparison with the static deflection. Table 3 shows the change in D_0 , D_1 with four values of applied voltage $V = 2, 6, 18, 24.4 \text{ V}$ for the microbeam with length $L = 210 \mu\text{m}$. When D_0 is very small, $D_1 \approx A$ is an approximate value of the vibration amplitude. However, as D_0 increases, D_1 becomes smaller and tends to a zero value, denoted as $D_{1,\text{ZP}}$. The point of the applied voltage at which D_1 equals zero is denoted as V_{ZP} , and called the zero-amplitude point (ZAP). From this point, the sign of D_1 is negative. The approximate maximum value of the response $u(\tau)$ is $u_{\text{max,approx}} = D_0 + |D_1|$. The graphs of the three curves, D_0 , D_1 , $u_{\text{max,approx}}$, are illustrated in figure 7 for the beam length $L = 210 \mu\text{m}$. The results obtained from the Runge–Kutta method for $u_{\text{max,RK}}$ are also depicted to show the accuracy of the EBM approach. In figure 7, as the applied voltage increases, it causes an increase in the electrostatic attraction between two electrodes; consequently, the maximum deflection of the beam increases and tends to a threshold point at which the electrostatic force becomes large, and overcomes the elastic force restored in the beam, called the pull-in instability point of the voltage, V_{PI} , as mentioned earlier. The approximation of the ZAP, V_{ZP} , is found to be $V_{\text{ZP}} = 18.4 \text{ V}$. (In

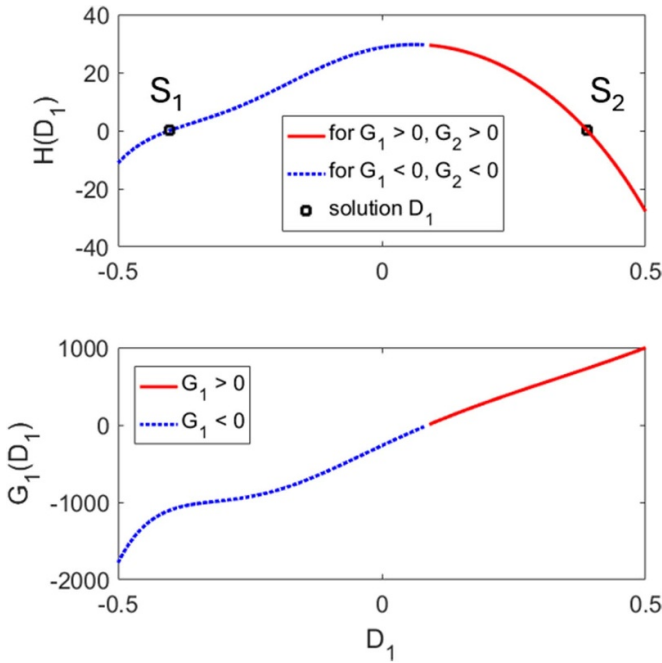


Figure 8. Graphs of the functions $H(D_1)$, $G(D_1)$ for conditions of solution existence of the EBM for the initial conditions $u(0) = A = 0.4$ at the applied voltage $V = 2$ (V) ($L = 210 \mu\text{m}$).

numerical calculation, the amplitude of vibration at $V_{ZP} = 18.4$ V is relatively small, nearly zero.) On the physical point of safety, the range of the applied voltage from ZAP V_{ZP} to the pull-in instability point V_{PI} can be considered as a warning domain of the voltage.

It can be observed from equation (25) that because $\omega^2 > 0$, $k^2 > 0$ leads to the conditions $G_1 > 0$, $G_2 > 0$. Therefore, in the process of finding D_1 from the algebraic equation (28), only solution D_1 that satisfies the conditions $G_1 > 0$, $G_2 > 0$ is accepted. For illustration, figure 8 plots the graphs of two functions, $H(D_1)$ from equation (28) and $G_1(D_1)$ from equation (25), depending on different values of D_1 for the beam length $L_1 = 210 \mu\text{m}$, at the applied voltage value $V = 2$ V and $A = 0.4$. The red solid line shows that the conditions $G_1 > 0$, $G_2 > 0$ are satisfied, whereas the blue dotted line presents the opposite case. In the process of solving equation (28) iteratively via the Newton–Raphson method, depending on the choice of the initial value of D_1 , we will obtain different solution values. Figure 8 shows that two different solution points S_1, S_2 can be found for D_1 when the initial value is chosen from -0.5 to 0.5 . The first solution point S_1 is located on the left branch of the blue dotted line, which is not the desired solution. On the other hand, at this value, i.e. $D_1 = -0.4031$, $D_0 = 0.8031$, making the maximum response $u_{max,approx}$ equal $1.2063 > 1$, has no physical meaning. Only the right solution point on the solid curve is the desired solution. The value of D_1 is found to be $D_1 = 0.3896$ and $D_0 = 0.0104$. The boundary point distinguishing the two regions of the obtained solutions is approximately $D_{1,boundary} = 0.09$. To see the convergence of the solution more clearly when choosing the initial conditions for D_1 , figure 9 portrays the basin of attraction in one dimension (1D) for the Newton–Raphson iteration method based on equation (28). Because there are only two

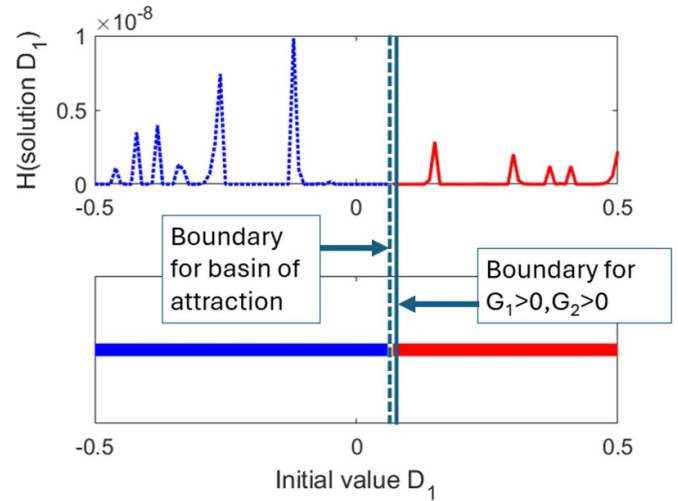


Figure 9. The basin of attraction in 1D for the Newton–Raphson iteration method in finding the solution D_1 for the initial value of D_1 from -0.5 to 0.5 (for the case: $A = 0.4$, $V = 2$ V, $L = 210 \mu\text{m}$).

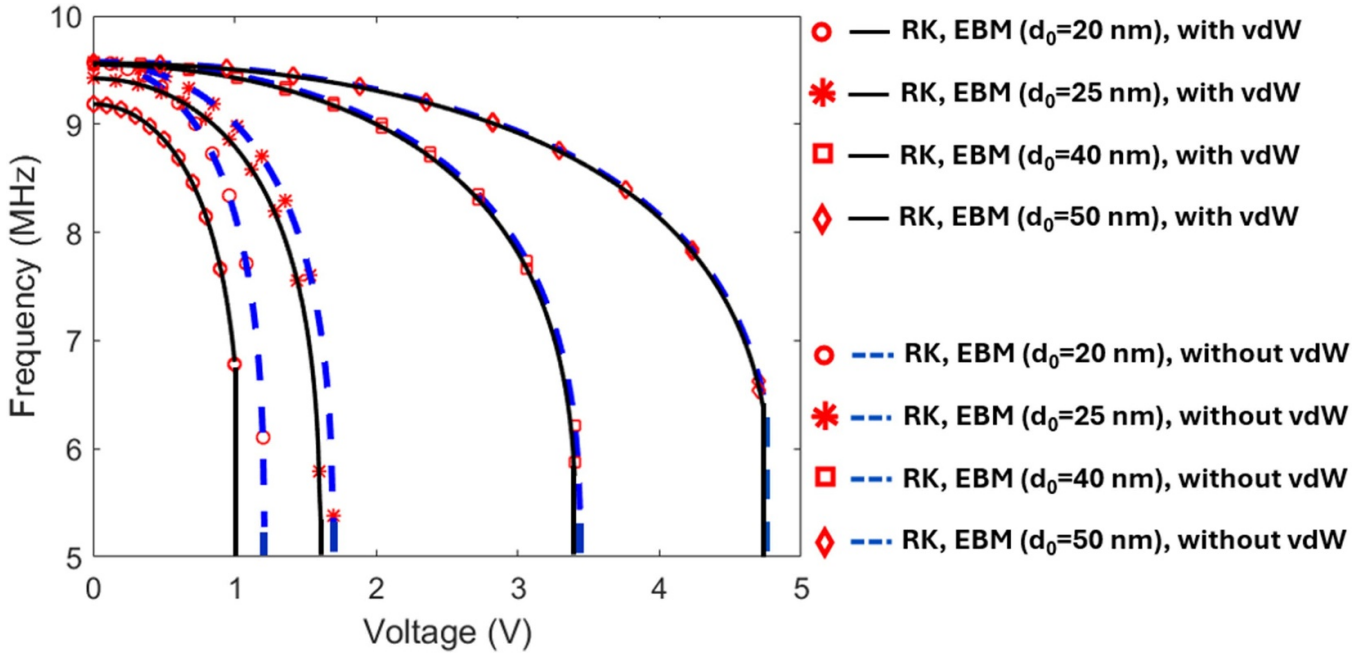
solution points S_1, S_2 in the considered interval $[-0.5, 0.5]$, we use two colors to display the basin of attraction. The blue color shows the initial conditions region, where the convergence of the solution will be toward point S_1 , and the red color shows the convergence leading to point S_2 . In the obtained solution region, the value of the function H is also displayed. In terms of numerical calculation, at the solution points, the value of H must be relatively small and almost zero. It is noted that the boundary point between the two convergence regions of the basin of traction may be different from the region where the conditions $G_1 > 0$, $G_2 > 0$ must be satisfied. In the situation under consideration, the boundary point between the two basin regions is found to be approximately 0.07 . Figure 9 illustrates these two boundary points. Thus, a narrow region $(0.07, 0.09)$ of the initial conditions D_1 exists, leading to converge to point S_2 , but does not satisfy the conditions $G_1 > 0$, $G_2 > 0$. This narrow region can change when calculating using different beam lengths at different voltages.

The above instructions and calculations reveal the following aspects:

- The accuracy obtained from the Newton–Raphson iteration method depends on the convergence criterion of the solution and the structures of the iteration. Here, the authors use the posterior convergence criterion, in which the error between two consecutive solutions in the iteration is less than a pre-selected tolerance.
- With the initial condition $u(0) = A$ having a large value, the solution calculation may be more sensitive to the choice of initial conditions for the Newton–Raphson iteration. Since the H curve is monotonic on the right branch, a suggestion for the initial conditions for the iteration is to choose D_1 positive, close to the value of A and smaller than A .
- The potential source of error affecting the accuracy of the elliptic balance solution remains an open question. One noticeable observation is that the EBM considered in this study is performing in the first-order approximation procedure using

Table 4. Geometrical and material parameters for the nanobeam [49].

L (nm)	b (nm)	h (nm)	E (GPa)	ν	ρ (kg m ⁻³)
5000	1000	100	80	0.42	19 300

**Figure 10.** Frequency curves of the nanobeam versus the applied voltage with different values of the gap distance d_0 in two situations: (1) the system model without van der Waals interaction; and (2) the system model with van der Waals interaction. The Runge–Kutta and EBM solutions are displayed for comparison.

the Jacobi elliptic function. A higher-order approximation is needed to increase the accuracy of the obtained solution. The characteristic of the method is that the number of unknowns is larger than the conventional harmonic balance method because of the presence of the modulus parameter. This leads to more difficulties in solving the obtained algebraic equations, especially in evaluating the initial conditions for the iterated solution. This is also a potential source of error and needs to be investigated in detail. In the present study, as a result of the analytical transformation, the number of equations and unknowns has been reduced, making numerical investigation more convenient and easier.

4.4. Response of the nanobeam with van der Waals interaction

In this subsection, we calculate the response of the nanobeam using the elliptic balance method. The dimensions and material parameters of the nanobeam are given in table 4. The axial residual stress is taken to be $\sigma_r = 10$ MPa. The Hamaker constant A_H for our calculation is $A_H = 44 \times 10^{-20}$ J. Here, the dimension of the nanobeam is reduced in comparison with the microbeam to show the effect of van der Waals forces. The van der Waals interaction is present when the gap d_0 between two electrodes is about several tens of nanometers. For numerical calculations, we illustrate four different values of d_0 : $d_0 = 20, 25, 40, 50$ nm. Two models are considered: (1) a model without van der Waals

interaction, called model (I); and (2) a model with van der Waals interaction, called model (II).

The behavior of the frequency curve of the vibration of the nanobeam is portrayed in figure 10. In both models, the pull-in instability phenomenon is observed. It can be seen that, at large gap distances d_0 , for example, $d_0 = 50$ nm, the difference between two frequency curves of two models is very small; consequently, the pull-in points found are nearly the same. This shows that the influence of van der Waals interaction force is negligible at the relatively large distance between the two electrodes. The observation is similar for $d_0 = 40$ nm, but the shift of the frequency curve begins to be observed more clearly in the two models. When decreasing the gap distance d_0 , the phenomenon of the frequency shift from model (I) to model (II) becomes obvious. For example, for $d_0 = 25$ nm, the frequency curve tends to the left, i.e. toward the decreasing direction of the voltage. This reflects the fact that model (II) demonstrates the presence of a relatively strong van der Waals interaction compared to model (I). Therefore, model (II) should be used instead of model (I) in the system at the nano-scale, similar to the reported results by Askari *et al* [49]. The results obtained from the EBM approach are nearly identical to those of the Runge–Kutta method for both models.

Figure 11 shows the correlation between the two forces, the electrostatic and the van der Waals interaction forces with different gap distances. It can be seen that at large gap distances, such as $d_0 = 50$ nm, the calculated value of the interaction forces is relatively small, at less than 1.4 mN for the electrostatic force and

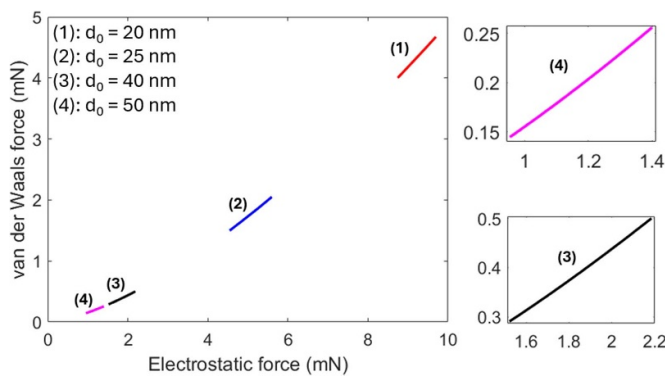


Figure 11. The relationship between van der Waals and electrostatic forces at 0.8 V with various values of the gap d_0 , $d_0 = 20, 25, 40$ and 50 nm.

less than 0.25 mN for the van der Waals force. In this case, the force ratio between the van der Waals to the electrostatic force is $0.25/1.4 = 0.18$. For $d_0 = 40$ nm, this value increases to $0.5/2.2 = 0.23$ and gradually increases when $d_0 = 20$ μm up to 0.5.

5. Conclusions and remarks

Finding exact solutions of a nonlinear system is relatively difficult. Therefore, finding an approximation method with acceptable solution accuracy is a viable alternative. In this study, we have dealt with the problem of nonlinear free vibration of micro/nano-beams under the influence of electrostatic and van der Waals interactions using the EBM. The research results are shown in the following main points:

- The obtained approximate solution gives a highly accurate result in comparison with that received from the numerical method using the Runge–Kutta algorithm and from other approximation methods, such as the Hamiltonian approach and the HAM. The high precision obtained allows us to find that the frequency approximation and pull-in instability points become more reliable. The error found is only about less than 0.1%, lower than other methods known to the authors.
- We point out the influence of van der Waals interaction forces on the response of the system, and this force has a relatively large effect in nanoscale systems. Therefore, the system model at the nanoscale needs to take into account this interaction force when the gap distance between the two electrodes is at a few tens of nanometers.
- Some observations on the solution behavior of elliptic balance approximate solutions are also analyzed in detail. We have found a point, called the ZAP, during the change in the applied voltage to the system. This point is considered as the beginning of a process of increasing amplitude as the voltage continues to increase until the pull-in instability phenomenon occurs. Boundaries for the existence of a solution for coefficients in elliptic balance presentation and for a 1D basin of attraction of the Newton–Raphson iteration method are shown. They are considered as a guide for choosing initial values for the iteration to obtain a physically meaningful desired solution.

It can be seen that the improved accuracy of the solution has an important significance in determining the frequency and pull-in instability point. Some points that can be highlighted are: (1) ensuring performance: accurate frequency helps ensure that devices operate within the desired frequency range. In many N/MEMS applications, such as accelerometers, pressure sensors or frequency filters, the correct operating frequency is essential for proper device function and accurate data output. (2) Optimization of design: accurate frequency allows engineers to design N/MEMS systems with greater precision. This can optimize the size, shape and materials of the components, improving the overall performance and reliability of the device. (3) Reducing sensitivity to environmental factors: factors such as temperature, pressure and humidity can affect the operating frequency of N/MEMS systems. The determination of accurate frequency and understanding the influencing factors help in designing mechanisms to adjust or protect against environmental impacts. (4) Designing and analyzing vibration characteristics: in N/MEMS designs, especially those related to mechanical vibration, accurately determining the frequency helps engineers analyze and predict the vibration characteristics of the system, avoiding unwanted phenomena like resonance. These important points may promote the use of accurate results from this study or future studies of N/MEMS structures. Furthermore, to confirm the effectiveness of the EBM, more systems need to be considered, such as resonators, micro-mirrors, actuators, optical switches, nanoscale force sensors, and nanoscale optical devices, and related nonlinear interactions such as electrostatic, van der Waals forces, fringing effects and Casimir force. This is a promising method for strongly nonlinear systems in the field of micro/nano-sized structures.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

The second author was supported by a grant with code T23-16 from the Hanoi University of Mining and Geology.

ORCID iD

Nguyen Nhu Hieu  <https://orcid.org/0000-0002-1512-3391>

References

- [1] Lyshevski S E 2002 *MEMS and NEMS Systems, Devices, and Structures* (CRC Press)
- [2] Magrab E B 2012 *Vibrations of Elastic Systems, with Applications to MEMS and NEMS* (Springer)
- [3] Kim S H, Dugger M T and Mittal K L 2010 *Adhesion Aspects in MEMS/NEMS* (CRC Press)
- [4] Sedighi H M 2014 Size-dependent dynamic pull-in instability of vibrating electrically actuated microbeams based on the strain gradient elasticity theory *Acta Astronaut.* **95** 111–23
- [5] Batra R C, Porfiri M and Spinello D 2008 Vibrations and pull-in instabilities of microelectromechanical von Kármán

- elliptic plates incorporating the Casimir force *J. Sound Vib.* **315** 939–60
- [6] Nathanson H C, Newell W E, Wickstrom R A and Davis J R 1967 The resonant gate transistor *IEEE Trans. Electron Dev.* **14** 117–33
- [7] Taylor G I 1968 The coalescence of closely spaced drops when they are at different electric potentials *Proc. R. Soc. A* **306** 423–34
- [8] Pelesko J A and Bernstein D H 2003 *Modeling MEMS and NEMS* (Chapman & Hall CRC)
- [9] Koochi A and Abadyan M 2020 *Nonlinear Differential Equations in Micro/nano Mechanics: Application in Micro/nano Structures and Electromechanical Systems* (Elsevier Ltd)
- [10] Mohammadian M 2024 From periodic to pseudo-periodic motion and pull-in instability of the MWCNT actuator in the vicinity of the graphite sheets *Chin. J. Phys.* **90** 557–71
- [11] Zand M M, Ahmadian M T and Rashidian B 2009 Semi-analytic solutions to nonlinear vibrations of microbeams under suddenly applied voltages *J. Sound Vib.* **325** 382–96
- [12] Tilmans H A C and Legtenberg R 1994 Electrostatically driven vacuum-encapsulated polysilicon resonators: part II. Theory and performance *Sens. Actuators A* **45** 67–84
- [13] Rashidi H, Rahimi Z and Sumelka W 2018 Effects of the slip boundary condition on dynamics and pull-in instability of carbon nanotubes conveying fluid *Microfluid. Nanofluidics* **22** 131
- [14] Anjum N, H H J, He C H and Gepreel K A 2023 Variational iteration method for prediction of the pull-in instability condition of micro/nanoelectromechanical systems *Phys. Mesomech.* **26** 241–50
- [15] Zarezadeh E, Eshaghi J and Barati A 2023 Static pull-in analysis of the cantilever and clamped FG-microswitches in presence of the longitudinal magnetic field based on the modified couple stress theory *Eur. Phys. J. Plus* **138** 524
- [16] Zhang Y and Zhao Y P 2006 Numerical and analytical study on the pull-in instability of micro-structure under electrostatic loading *Sens. Actuators A* **127** 366–80
- [17] Hu Y C and Wei C S 2007 An analytical model considering the fringing fields for calculating the pull-in voltage of micro curled cantilever beams *J. Micromech. Microeng.* **17** 61–67
- [18] Park S K and Gao X L 2006 Bernoulli–Euler beam model based on a modified couple stress theory *J. Micromech. Microeng.* **16** 2355
- [19] Vo D, Suttakul P, Rungamornrat J and Nanakorn P 2022 Static analysis of planar arbitrarily curved microbeams with the modified couple stress theory and Euler–Bernoulli beam model *Appl. Math. Modelling* **112** 358–90
- [20] Roudbari M A, Jorshari T D, C L, Ansari R, Kouzani A Z and Amabili M 2022 A review of size-dependent continuum mechanics models for micro- and nano-structures *Thin-Walled Struct.* **170** 108562
- [21] Rahaeifard M, Kahrobaiyan M K, Asghari M and Ahmadian M T 2011 Static pull-in analysis of microcantilevers based on the modified couple stress theory *Sens. Actuators A* **171** 370–4
- [22] Ghayesh M H, Farokhi H and Amabili M 2013 Nonlinear behaviour of electrically actuated MEMS resonators *Int. J. Eng. Sci.* **71** 137–155
- [23] Dai H L, Wang Y K and Wang L 2015 Nonlinear dynamics of cantilevered microbeams based on modified couple stress theory *Int. J. Eng. Sci.* **94** 103–12
- [24] Attia M A and Mohamed S A 2017 Nonlinear modeling and analysis of electrically actuated viscoelastic microbeams based on the modified couple stress theory *Appl. Math. Modelling* **41** 195–222
- [25] Farokhi H and Ghayesh M H 2017 Nonlinear thermo-mechanical behaviour of MEMS resonators *Microsyst. Technol.* **23** 5303–15
- [26] Askari A R and Awrejcewicz J 2023 Modified couple stress flexural–flexural quasi-static pull-in analysis of large deformable cantilever-based micro-gyroscopes *Commun. Nonlinear Sci. Numer. Simul.* **117** 106933
- [27] Rhaeifard M and Ahmadian M T 2015 On pull-in instability of microcantilevers *Int. J. Eng. Sci.* **87** 23–31
- [28] Jia X L, Yang J, Kitipornchai S and Lim C W 2012 Pull-in instability and free vibration of electrostatically actuated poly-SiGe graded microbeams with a curved ground electrode *Appl. Math. Modelling* **36** 1875–84
- [29] Tahani M and Askari A R 2014 Accurate electrostatic and van der Waals pull-in prediction for fully clamped nano/micro-beams using linear universal graphs of pull-in instability *Physica E* **63** 151–9
- [30] Wang K F, Kitamura T and Wang B 2015 Nonlinear pull-in instability and free vibration of micro/nano-scale plates with surface energy—a modified couple stress theory model *Int. J. Mech. Sci.* **99** 288–96
- [31] Attia M A and Mohamed S A 2019 Coupling effect of surface energy and dispersion forces on nonlinear size-dependent pull-in instability of functionally graded micro-/nanoswitches *Acta Mech.* **230** 1181–216
- [32] Zhang W M, Yan H, Peng Z K and Meng G 2014 Electrostatic pull-in instability in MEMS/NEMS: a review *Sens. Actuators A* **214** 187–218
- [33] Hajjaj A Z, Jaber N, Ilyas S, Alfossail F K and Younis M I 2020 Linear and nonlinear dynamics of micro and nano-resonators: review of recent advances *Int. J. Non-Linear Mech.* **119** 103328
- [34] Rotkin S V 2002 Analytical calculations for nanoscale electromechanical systems *Microfabricated Systems and MEMS VI: Proc. Int. Symp.* vol 6 pp 90–97
- [35] Ramezani A, Aria A and Javad A 2006 Influence of van der Waals force on the pull-in parameters of cantilever type nanoscale electrostatic actuators *Microsyst. Technol.* **12** 1153–61
- [36] Aykol M, Hou B, Dhall R, Chang S W, Branham W, Qiu J and Cronin S B 2014 Clamping instability and van der Waals forces in carbon nanotube mechanical resonators *Nano Lett.* **14** 2426–30
- [37] Bhojwala V M and Vakharia D P 2017 Effect of van der Waals force on pull-in voltage, frequency tuning and frequency stability of NEMS devices *Microsyst. Technol.* **23** 1255–67
- [38] Taghavi N and Nahvi H 2013 Pull-in instability of cantilever and fixed-fixed nano-switches *Eur. J. Mech. A* **41** 123e133
- [39] Yazdanpanahi E, Noghrehabadin A and Ghalambaz M 2014 Pull-in instability of electrostatic doubly clamped nano actuators: introduction of a balanced liquid layer *Int. J. Non-Linear Mech.* **58** 128–38
- [40] Moradweysi P, Ansari R, Hosseini K and Sadeghi F 2018 Application of modified Adomian decomposition method to pull-in instability of nano-switches using nonlocal Timoshenko beam theory *Appl. Math. Modelling* **54** 594–604
- [41] Mirkalantari S A, Hashemian M, Eftekhari S A and Toghraie D 2017 Pull-in instability analysis of rectangular nanoplate based on strain gradient theory considering surface stress effects *Physica B* **519** 1–14
- [42] Dehkordi M F, Beni Y T, Dashtaki P M and Vanani S M F 2023 Effect of flexoelectricity on the pull-in instability of beam-type NEMS *Eng. Anal. Bound. Elem.* **155** 87–107
- [43] Mikhaseva G, Radib E and Misnik V 2024 Modeling pull-in instability of CNT nanotweezers under electrostatic and van der Waals attractions based on the nonlocal theory of elasticity *Int. J. Eng. Sci.* **195** 104012

- [44] He J H 2000 Some asymptotic methods for strongly nonlinear equations *Int. J. Modern Phys. B* **2006** 51–71
- [45] Maluf N and Williams K 2004 *An Introduction to MEMs Engineering* (Artech House Publishers)
- [46] Gad-el-hak M 2005 *The MEMS Handbook Introduction & Fundamentals* (CRC Press)
- [47] Esmailzadeh E, Younesian D and Askari H 2019 *Analytical Methods in Nonlinear Oscillations: Solid Mechanics and Its Applications* (Springer)
- [48] Askari A R and Tahani M 2012 Analytical approximations to nonlinear vibration of a clamped nanobeam in presence of the Casimir force *Int. J. Aerosp. Lightweight Struct.* **2** 317–34
- [49] Askari A R, Tahani M and Moeenfarid H 2017 A frequency criterion for doubly clamped beam-type N/MEMS subjected to the van der Waals attraction *Appl. Math. Modelling* **41** 650–66
- [50] Ismail G M and Cveticanin L 2021 Higher order Hamiltonian approach for solving doubly clamped beam type N/MEMS subjected to the van der Waals attraction *Chin. J. Phys.* **72** 69–77
- [51] Shamsmohammadi N, Samadi H, Rahimzadeh M, Asadi Z and Ganji D D 2023 Nano/micro-beam deflections: investigation of subjected forces and applications *Phys. Open* **17** 100191
- [52] Nayfeh A H 2000 *Perturbation Methods* (Wiley)
- [53] Krack M and Gross J 2019 *Harmonic Balance for Nonlinear Vibration Problems* (Springer)
- [54] Rafiee M, Liu X F, He X Q and Kitipornchai S 2014 Geometrically nonlinear free vibration of shear deformable piezoelectric carbon nanotube/fiber/polymer multiscale laminated composite plates *J. Sound Vib.* **333** 3236–51
- [55] Kuang J H and Chen J H 2005 Adomian decomposition method used for solving nonlinear pull-in behavior in electrostatic micro-actuators *Math. Comput. Modelling* **41** 1479–91
- [56] Tang W, Anjum N and He J H 2023 Variational iteration method for the nanobeams-based N/MEMS system *Methods X* **11** 102465
- [57] Garcia-Margallo J and Bejarano J D 1990 Generalized Fourier series and limit cycles of generalized van der Pol oscillators *J. Sound Vib.* **1990** 453–66
- [58] Mickens R E 2010 *Truly Nonlinear Oscillations: Harmonic Balance, Parametric Expansions, Iteration, and Averaging Methods* (World Scientific)
- [59] Rand R H 1990 Using computer algebra to handle elliptic functions in the method of averaging *Symbolic Computations and Their Impact on Mechanics* vol 205, ed A K Noor, I Elishakoff and G Hulbert (American Society of Mechanical Engineers) pp 311–26
- [60] Belhaq M and Lakrad F 2000 On the elliptic harmonic balance method for mixed parity non-linear oscillators *J. Sound Vib.* **233** 935–7
- [61] Elías-Zúñiga A 2013 Exact solution of the cubic-quintic Duffing oscillator *Appl. Math. Modelling* **37** 2574–9
- [62] Lenci S 2022 Exact solutions for coupled Duffing oscillators *Mech. Syst. Signal Process.* **165** 108299
- [63] Vakakis A F and Blanchard A 2018 Exact steady states of the periodically forced and damped Duffing oscillator *J. Sound Vib.* **413** 57–65
- [64] Kovacic I, Cveticanin L, Zukovic M and Rakaric Z 2016 Jacobi elliptic functions: a review of nonlinear oscillatory application problems *J. Sound. Vib.* **380** 1–36
- [65] Elías-Zúñiga A 2003 On the elliptic balance method *Math. Mech. Solids* **8** 263–79
- [66] Hsu C S 1960 On the application of elliptic functions in nonlinear forced oscillations *Q. Appl. Math.* **17** 393–407
- [67] Elías-Zúñiga A 2006 A general solution of the Duffing equation *Nonlinear Dyn.* **45** 227–35
- [68] Elías-Zúñiga A 2005 Application of Jacobian elliptic functions to the analysis of the steady-state solution of the damped Duffing equation with driving force of elliptic type *Nonlinear Dyn.* **42** 175–84
- [69] Elías-Zúñiga A, Rodríguez C A and Martínez-Romero O 2012 Application of the elliptic balance method to a nonlinear singular oscillator *Appl. Math. Comput.* **218** 11112–7
- [70] Wu W and Tang B 2020 An approximate method for solving force and displacement transmissibility of a geometrically nonlinear isolation system *Int. J. Non-Linear Mech.* **125** 103512
- [71] Banerjee A and Bera K K 2023 Wave propagation in mass-in-mass Duffing type non-linear metamaterial implementing Jacobi's elliptic balance method *Int. J. Non-Linear Mech.* **157** 104549
- [72] Gu Y 2001 Experimental determination of the Hamaker constants for solid–water–oil systems *J. Adhes. Sci. Technol.* **15** 1263–83
- [73] Leipholz H H E 1976 Use of Galerkin's method for vibration problems *Shock Vib. Dig.* **8** 3–18
- [74] Elishakoff I and Zingales M 2004 Convergence of Bubnov–Galerkin method exemplified *AIAA J.* **42** 1931–3
- [75] Repi S 2017 One hundred years of the Galerkin method *Comput. Methods Appl. Math.* **17** 351–7
- [76] Atkinson K E 1989 *An Introduction to Numerical Analysis* 2nd edn (Wiley)
- [77] Endre S and Davidm M 2003 *An Introduction to Numerical Analysis* (Cambridge University Press)
- [78] Otto S R and Denier J P 2005 *An Introduction to Programming and Numerical Methods in MATLAB* (Springer)