

IMA Journal of Mathematical Control and Information Page 1 of 14
doi:10.1093/imamci/dri000

Finite-time H_∞ control of linear singular fractional differential equations with time-varying delay

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[Received]

In this paper, we propose an efficient analytical approach based on fractional calculus and singularity value theory to designing the finite-time H_∞ controller for linear singular fractional differential equations with time-varying delay. By introducing new fractional-order H_∞ norm, the state feedback controller is designed to guarantee that the closed-loop system is singular, impulse-free and finite-time stable with prescribed H_∞ performance. New sufficient conditions for designing the H_∞ finite-time controller are presented. The results of this paper improve the corresponding ones of integer-order singular systems with time-varying delay. Finally, a numerical example demonstrates the validity and effectiveness of the proposed theoretical results.

Keywords: Fractional derivative, Finite-time control, Singular systems, Mittag-Leffler functions, Laplace transform, time-varying delays.

1. Introduction

During the past decades, the research on fractional differential equations (FDEs) has attracted increasing interest and has numerous applications in science and engineering. There has been a significant theoretical development in FDEs (Podlubny, 1999; Kilbas et al., 2006; Lakshmikantham and Vatsala, 2007; Kamal et al., 2015; Buslowicz, 2016; Wu and Baleanu, 2018). So far, there has been a growing research interest on the H_∞ control problem of dynamical systems (Francis, 1987; Petersen et al., 2000). The research activities for the problem have been developed for integer-order systems (Fridman and Shaked, 2003; Haidi and Mohamad, 2020; Zhang et al., 2018). For fractional-order systems, there are some results about H_∞ control for FDEs. In (Boukal et al., 2017) the authors studied H_∞ observer-based control of FDEs with gain parametrization. The authors of (Shen and Lam, 2014) designed state feedback H_∞ controllers of commensurate FDEs, meanwhile the paper (N'Doye et al, 2015) designed output feedback H_∞ controllers of Glucose-insulin FDEs without delays.

In practice, one concerns the boundedness of solutions during a prescribed time interval, which is called finite-time stability (FTS) (Donato, 1961; Amato et al., 2014). FTS involves dynamical systems whose solutions are bounded over finite time. Many significant results on the finite-time stability and control have been obtained in the last decade. For integer-order systems, paper (Guo et al. 2013) proposed significant criteria for designing finite-time H_∞ controllers of linear systems without delay by using Lyapunov function method and linear matrix inequality techniques. In (Zong et al., 2015, Sun and Zhao, 2021), the authors studied finite-time H_∞ control for switched nonlinear discrete systems with delays by applying Lyapunov-Krasovskii method. For fractional-order systems, the authors of (Liu et al., 2018, Thuan et al., 2020) investigated finite-time H_∞ control for FDEs without delays. On the other

hand, the singular (or differential-algebraic, implicit, descriptor) systems have received substantial interest from researchers due to their widespread practical applications in control engineering (Dai, 1989; Wu et al. 2013, Fridman, 2002). When studying stability and control of singular systems, an aspect to be considered is ensuring the admissibility (stability, impulse-free and regularity) of the system. By employing the singular value decomposition and Lyapunov function method, there are various significant results on the stability and control problem proposed in (Mahmoud, 2009; Osorio-Gordillo, 2018; Phat et al., 2020, Chen et al., 2021; Du et al., 2021) for integer-order singular systems with delays. For singular FDEs with delays, the authors of (Pang and Jiang, 2014, Zhang et al., 2014, Liu et al., 2017, Thanh et al., 2020) proposed delay-dependent conditions for the asymptotic and finite-time stability. It is worth noting that there are no results on finite-time H_∞ control of singular FDEs with delays, since the classical solutions and methodology of singular FDEs cannot be applied for these systems.

In this paper, we study the problem of finite-time H_∞ control for linear singular FDEs with interval time-varying delay. The contributions of our paper are summarized as follows.

- (i) The singular FDEs under consideration deals with interval nondifferentiable time-varying delay and normed bounded perturbations.
- (ii) A novel approach based on the fractional calculus (Caputo derivative, Laplace transform, the Mittag-Leffler function) and the singularity value theory is proposed to solve the finite-time H_∞ control problem.
- (iii) By introducing a new fractional-order H_∞ norm, the delay-dependent conditions are derived and presented in the form of tractable matrix inequalities.
- (iv) Comparing with the existing works, the proposed results can be applied to more general FDEs with delays including the works of (Zhang et al., 2014, Shen & Lam, 2014, Mathiyalgan and Balakchandran, 2016, Liu et al., 2018, Osorio-Gordillo et al., 2018, Thuan et al., 2020).

The paper is organized as follows. Section 2 introduces basic definitions of Captupo derivative with its properties and the control problem to be considered. Mail result on the design of finite-time H_∞ controllers is presented in Section 3 with a numerical example and simulations. Finally, Section 4 summarizes the conclusions.

2. Preliminaries

The following notations are used throughout this paper. $\mathbb{C}$ denotes the complex spaces; $\mathbb{R}^n$ denotes the n -dimensional Euclidean space; $\mathbb{R}^{n \times m}$ stands for the space of all $(n \times m)$ - matrices; $\lambda(A)$ denotes the set of all eigenvalues of A ; $\lambda_{\min}(A) = \min\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$; A is positive definite ($A > 0$) if $(Ay, y) > 0$ for all $y \neq 0$; $L^1[0, b]$ denotes the space of integrable functions on $[0, b]$; ΔV denotes the gradient of vector function $V(\cdot)$, $[a]$ denotes the integer part of a .

In order to describe the problem, we will recall some basic definitions about fractional calculus and several useful lemmas of (Podlubny, 1999; Kilbas et al., 2006). For $\alpha \in (0, 1)$ and $x \in L^1[0, T]$, the Riemann-Liouville integral and the Riemann-Liouville derivative are respectively defined by

$$\begin{aligned}\mathcal{I}^\alpha x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \\ \mathcal{D}_R^\alpha x(t) &= \frac{d}{dt} (\mathcal{I}^{1-\alpha} x(t)),\end{aligned}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$, $\operatorname{Re}(s) > 0$ is the gamma function. The Caputo derivative is then defined as

$$\mathcal{D}_C^\alpha x(t) = \mathcal{D}_R^\alpha (x(t) - x(0)).$$

Recall the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and the Laplace transform $\mathcal{L}[x(t)](s)$ defined respectively by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \mathcal{L}[x(t)](s) = \int_0^{\infty} e^{-st} x(t) dt.$$

For $\beta = 1$, denote $E_\alpha(z) := E_{\alpha,1}(z)$.

LEMMA 2.1 (Kilbas et al., 2006). For $\alpha > 0, z \geq 0$, the Mittag-Leffler function $E_\alpha(z)$ is an increasing function and $E_\alpha(z) \geq 1$.

LEMMA 2.2 (Kilbas et al., 2006) Let $x, y \in L^1[0, T], \alpha \in (0, 1), h > 0$, we have

- (a) $\mathcal{L}[\mathcal{D}_C^\alpha x(t)](\xi) = \xi^\alpha \mathcal{L}[x(t)](\xi) - \xi^{\alpha-1} x(0)$.
- (b) $\mathcal{L}[t^{\alpha h} E_\alpha^{(k)}(ht^\alpha)](\xi) = \frac{k! \xi^{\alpha-1}}{(\xi^\alpha - h)^{k+1}}, k = 1, 2, \dots, \operatorname{Re}(\xi) > h^{1/\alpha}$.
- (c) $\mathcal{L}[x * y(t)](\xi) = \mathcal{L}[x(t)](\xi) \cdot \mathcal{L}[y(t)](\xi)$,

where the convolution of $x(t)$ and $y(t)$ is defined by $x * y(t) = \int_0^t x(t - \tau) y(\tau) d\tau$.

LEMMA 2.3 (Trinh & Tuan, 2018) Let $x(t)$ be a continuous function, $V(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a convex and differentiable function such that $V(0) = 0$. We have

$$\mathcal{D}_C^\alpha V(x(t)) \leq \langle \Delta V(x(t)), \mathcal{D}_C^\alpha x(t) \rangle, \quad t \geq 0.$$

Consider the following singular FDEs with time-varying delay:

$$\begin{cases} E \mathcal{D}_C^\alpha x(t) &= Ax(t) + Dx(t - \tau(t)) + Bu(t) + W\omega(t), \\ z(t) &= Cx(t) + Hx(t - \tau(t)), \\ x(s) &= \varphi(s), \quad s \in [-\tau_2, 0], \end{cases} \quad (2.1)$$

where $0 < \alpha < 1; x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $\omega(t) \in \mathbb{R}^p$ is the disturbance, E is a singular matrix: $\operatorname{rank} E = r < n; A, D, C, H \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, W \in \mathbb{R}^{n \times p}$ are given constant matrices, $\varphi(\cdot)$ is the continuous initial function, the delay function $\tau(t)$ satisfies

$$\exists \tau_1, \tau_2 > 0: \quad 0 < \tau_1 \leq \tau(t) \leq \tau_2, \quad t \geq 0;$$

the disturbance $\omega(t)$ satisfies

$$\exists d > 0: \quad \sup_{t \geq 0} \|\omega(t)\|^2 \leq d, \quad t \geq 0. \quad (2.2)$$

As in (Dai, 1989) we recall that system (2.1) is regular if $\det(sE - A)$ is not identically zero and is impulse-free if $\deg(\det(sE - A)) = \operatorname{rank} E$, for some $c \in \mathbb{C}$.

DEFINITION 2.1 For scalars $c_1 > 0, c_2 > 0, T > 0$, system (2.1) is FTS w.r.t. (c_1, c_2, T) if it is regular, impulse-free and the following relation holds:

$$\sup_{s \in [-\tau_2, 0]} \|\varphi(s)\|^2 \leq c_1 \implies \|x(t)\|^2 \leq c_2, \quad t \in [0, T].$$

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Let us introduce the fractional-order norm of $f(t)$ defined by

$$\|f\|_\alpha := \left\{ \sup_{s \in [0, T]} \mathcal{I}^\alpha \|f(s)\|^2 \right\}^{1/2}.$$

It is notable that $\|f\|_\alpha$ defines a norm in $L^1[0, T]$, and for $\alpha = 1$ this norm is reduced to the L_2 norm of $f(t)$ because of

$$\sup_{s \in [0, T]} \mathcal{I}^\alpha \|f(s)\|^2 = \sup_{s \in [0, T]} \int_0^s \|f(\tau)\|^2 d\tau = \int_0^T \|f(\tau)\|^2 d\tau.$$

DEFINITION 2.2 For scalars $c_1 > 0, c_2 > 0, T > 0, \gamma > 0$, the finite-time H_∞ control problem for system (2.1) is solvable if there exist a number $c_0 > 0$ and a control $u(t) = Lx(t)$ such that the resulting closed-loop system is FTS w.r.t. (c_1, c_2, T) and

$$\sup \frac{\|z(s)\|_\alpha^2}{c_0 \|\varphi\|^2 + \|\omega(s)\|_\alpha^2} \leq \gamma, \quad (2.3)$$

where the supremum is taken over all non-zero initial conditions $\varphi(\cdot)$ and disturbances $\omega(t)$ satisfying (2.2).

3. Finite-time H_∞ control design

This section presents delay-dependent criteria for designing the finite-time H_∞ controllers. Consider singular system (2.1), where the singular matrix E is, without of generality as in (Fridman, 2002), assumed that $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. By using the feedback controller $u(t) = Lx(t)$, where the matrix $L \in R^{m \times n}$ will be designed, the closed-loop system of (2.1) has the form

$$\begin{cases} E \mathcal{D}_C^\alpha x(t) &= \mathbb{A}x(t) + Dx(t - \tau(t)) + W\omega(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau_2, 0], \end{cases} \quad (3.1)$$

where $\mathbb{A} = A + BL$. Setting

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix}; \quad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}; \quad W\omega(t) = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix},$$

under the state transformation $x(t) = [x_1(t), x_2(t)]$, $x_1(t) \in R^r$, $x_2(t) \in R^{n-r}$, the closed loop system (3.1) has the form of fractional differential-algebraic equations

$$\begin{cases} \mathcal{D}_C^\alpha x_1(t) &= \mathbb{A}_{11}x_1(t) + \mathbb{A}_{12}x_2(t) + D_{11}x_1(t - \tau(t)) + D_{12}x_2(t - \tau(t)) + \omega_1(t), \\ 0 &= \mathbb{A}_{21}x_1(t) + \mathbb{A}_{22}x_2(t) + D_{21}x_1(t - \tau(t)) + D_{22}x_2(t - \tau(t)) + \omega_2(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau_2, 0]. \end{cases} \quad (3.2)$$

For given matrices E, A, B, W and matrices A_{22}^{-1}, Y , non-singular matrix $P \in R^{n \times n}$ to be defined, we introduce the following notations.

$$\bar{E} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \quad \gamma_0 = \frac{\gamma}{1 + 2\tau_2 E_\alpha(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \frac{T^\alpha}{\alpha}},$$

$$\begin{aligned}
\beta &= \max\{\|\mathbb{A}_{22}^{-1}\mathbb{A}_{21}\|, \|\mathbb{A}_{22}^{-1}D_{21}\|, \|\mathbb{A}_{22}^{-1}D_{22}\|\}, \\
\beta_1 &= \lambda_{\max}(P^{-1}E) \sum_{j=0}^{[T/\tau_1]+1} (E_\alpha(\tau_2 T^\alpha) - 1)^j E_\alpha(\tau_2 T^\alpha), \\
\beta_2 &= \gamma_0 E_\alpha(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]+1} (E_\alpha(\tau_2 T^\alpha) - 1)^j \frac{T^\alpha}{\alpha}, \\
\beta_3 &= 1 + \frac{\beta_1}{\lambda_{\min}(P_1)}, \beta_4 = \frac{\beta_2}{\lambda_{\min}(P_1)}, \beta_5 = \max_{i=0,1,2,\dots,[T/\tau_1]} \sum_{k=0}^i \beta^k, \beta_6 = \max_{i=0,1,2,\dots,[T/\tau_1]} \beta^{i+1}, \\
\beta_7 &= \|\mathbb{A}_{22}^{-1}\| \|W\| \sqrt{d}, \quad \zeta = 2\beta \sqrt{\beta_3 c_1 + \beta_4} + \beta_7, \\
\mathbb{E}_{11} &= AP^\top + PA^\top + BY + (BY)^\top - \tau_2 EP^\top + \frac{4}{\gamma_0} WW^\top + \bar{E}AP^\top + P(\bar{E}A)^\top \\
&\quad + \bar{E}BY + (\bar{E}BY)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \\
\mathbb{E}_{12} &= DP^\top + \bar{E}DP^\top + P(\bar{E}A)^\top + (\bar{E}BY)^\top, \mathbb{E}_{13} = PA^\top + (BY)^\top, \\
\mathbb{E}_{22} &= -\tau_2 EP^\top + \frac{4}{\gamma_0} WW^\top + \bar{E}DP^\top + P(\bar{E}D)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \mathbb{E}_{33} = -P^\top - P + \frac{4}{\gamma_0} WW^\top.
\end{aligned}$$

THEOREM 3.1 For scalars $\gamma > 0, c_1 > 0, c_2 > 0, T > 0$, problem of finite-time H_∞ control for system (2.1) is solvable if there exist a non-singular matrix P and a matrix Y such that the following conditions hold:

$$EP^\top = PE^\top \geq 0, \quad (3.3)$$

$$\begin{pmatrix}
\mathbb{E}_{11} & \mathbb{E}_{12} & \mathbb{E}_{13} & PC^\top & 0 \\
* & \mathbb{E}_{22} & PD^\top & 0 & PH^\top \\
* & * & \mathbb{E}_{33} & 0 & 0 \\
* & * & * & -0.5I & 0 \\
* & * & * & * & -0.5I
\end{pmatrix} < 0, \quad (3.4)$$

$$\frac{\beta_2 c_1 + \beta_3}{\lambda_{\min}(P_1)} + (\zeta \beta_5 + \beta_6 \sqrt{c_1})^2 \leq c_2. \quad (3.5)$$

The state feedback control is defined by $u(t) = Y(P^{-1})^\top x(t)$, $t \in [0, T]$.

Proof.

Step 1. Regularity and impulse-free. Setting $P^{-1} = \begin{pmatrix} P_1 & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ and using condition (3.3), we have

$$\begin{aligned}
P^{-1}E &= P^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ P_{21} & 0 \end{pmatrix} \geq 0, \\
E^\top [P^{-1}]^\top &= \begin{pmatrix} P_1^\top & P_{21}^\top \\ 0 & 0 \end{pmatrix} \geq 0,
\end{aligned}$$

and hence

$$P_{21} = 0, P_1 = P_1^\top \geq 0, P^{-1}E = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, P^{-1} = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix}. \quad (3.6)$$

Since matrix P is nonsingular and $P^{-1} = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix}$, from (3.6) it follows that $\det(P_1) \neq 0, P_1 > 0$.

Moreover, by the Schur complement lemma, the LMI condition (3.4) gives

$$AP^\top + PA^\top + BY + (BY)^\top - \tau_2 EP^\top + \bar{E}AP^\top + P(\bar{E}A)^\top + \bar{E}BY + (\bar{E}BY)^\top < 0,$$

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and hence

$$\begin{aligned} & P^{-1}(A + BY(P^{-1})^\top) + [A + BY(P^{-1})^\top]^\top (P^{-1})^\top - \tau_2 P^{-1}E \\ & + P^{-1}\bar{E}A + (\bar{E}A)^\top (P^{-1})^\top + P^{-1}\bar{E}BY(P^{-1})^\top + [P^{-1}\bar{E}BY(P^{-1})^\top]^\top < 0. \end{aligned}$$

Letting $L = Y(P^{-1})^\top$ gives

$$P^{-1}\mathbb{A} + \mathbb{A}^\top (P^{-1})^\top - \tau_2 P^{-1}E + P^{-1}\bar{E}\mathbb{A} + (P^{-1}\bar{E}\mathbb{A})^\top < 0. \quad (3.7)$$

Besides, we describe the matrices $P^{-1}\mathbb{A}$, $P^{-1}E$, $P^{-1}\bar{E}\mathbb{A}$ as follows

$$\begin{aligned} P^{-1}\mathbb{A} &= \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix} = \begin{pmatrix} P_1\mathbb{A}_{11} + P_{12}\mathbb{A}_{21} & P_1\mathbb{A}_{12} + P_{12}\mathbb{A}_{22} \\ P_{22}\mathbb{A}_{21} & P_{22}\mathbb{A}_{22} \end{pmatrix}, \\ P^{-1}E &= \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, \\ P^{-1}\bar{E}\mathbb{A} &= P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \mathbb{A} = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix} = \begin{pmatrix} P_{12}\mathbb{A}_{21} & P_{12}\mathbb{A}_{22} \\ P_{22}\mathbb{A}_{21} & P_{22}\mathbb{A}_{22} \end{pmatrix}. \end{aligned}$$

Then, combining with the derived inequality (3.7) we obtain that

$$(P_{22})\mathbb{A}_{22} + \mathbb{A}_{22}^\top (P_{22})^\top < 0,$$

which gives $\det(\mathbb{A}_{22}) \neq 0$ and then the system is regular and impulse-free.

Step 2. Finite-time stability. Consider the functional $\mathbb{V}(x(t)) = x(t)^\top P^{-1}Ex(t)$. Since $\mathbb{V}(\cdot)$ is a non-negative convex function satisfying condition of Lemma 2.3, we then can estimate the Caputo derivative of $\mathbb{V}(x(t))$ as

$$\mathcal{D}_C^\alpha(\mathbb{V}(x(t))) \leq 2x(t)^\top P^{-1}E\mathcal{D}_C^\alpha(x(t)), \quad t \geq 0.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) &\leq 2x(t)^\top P^{-1} \left(Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t) \right) \\ &\leq 2x(t)^\top P^{-1} \left(Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t) \right) \\ &\quad - \tau_2 x(t - \tau(t))^\top P^{-1}Ex(t - \tau(t)) + \tau_2 V(x(t - \tau(t))) \\ &\quad - \tau_2 x(t)^\top P^{-1}Ex(t) + \tau_2 V(x(t)) + 2\|Cx(t)\|^2 \\ &\quad + 2\|Hx(t - \tau(t))\|^2 - 2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2. \end{aligned} \quad (3.8)$$

Multiplying left both sides of (2.1) by $2(E\mathcal{D}_C^\alpha x(t))^\top P^{-1}$, $2x(t)^\top P^{-1}\bar{E}$, $2x(t - \tau(t))^\top P^{-1}\bar{E}$, we get

$$\begin{aligned} 0 &= -2(E\mathcal{D}_C^\alpha x(t))^\top P^{-1}E\mathcal{D}_C^\alpha x(t) + 2(E\mathcal{D}_C^\alpha x(t))^\top P^{-1}[Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t)], \\ 0 &= -2x(t)^\top P^{-1}\bar{E}E\mathcal{D}_C^\alpha x(t) + 2x(t)^\top P^{-1}\bar{E}[Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t)], \\ 0 &= -2x(t - \tau(t))^\top P^{-1}\bar{E}E\mathcal{D}_C^\alpha x(t) + 2x(t - \tau(t))^\top P^{-1}\bar{E}[Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t)]. \end{aligned} \quad (3.9)$$

Using the following inequalities

$$\begin{aligned} \frac{4}{\gamma_0} x^\top P^{-1} \bar{E} W (P^{-1} \bar{E} W)^\top x + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2x^\top P^{-1} \bar{E} W \omega(t), \\ \frac{4}{\gamma_0} x(t)^\top P^{-1} W (P^{-1} W)^\top x(t) + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2x(t)^\top P^{-1} W \omega(t), \\ \frac{4}{\gamma_0} (E \mathcal{D}_C^\alpha x(t))^\top P^{-1} W (P^{-1} W)^\top (E \mathcal{D}_C^\alpha x(t)) + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2(E \mathcal{D}_C^\alpha x(t))^\top P^{-1} W \omega(t), \\ \frac{4}{\gamma_0} x(t - \tau(t))^\top P^{-1} \bar{E} W (P^{-1} \bar{E} W)^\top x(t - \tau(t)) + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2x(t - \tau(t))^\top P^{-1} \bar{E} W \omega(t), \end{aligned}$$

and from (3.8)-(3.9) it follows that

$$\begin{aligned} \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) &\leq \zeta(t)^\top \Omega \zeta(t) + \tau_2 \mathbb{V}(x(t - \tau(t))) + \tau_2 \mathbb{V}(x(t)) \\ &\quad - 2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2 + \gamma_0 \|\omega(t)\|^2, \end{aligned} \quad (3.10)$$

where $\zeta(t) = [x(t), x(t - \tau(t)), E \mathcal{D}_C^\alpha x(t)]$, $\Omega = [\Omega_{ij}]_{3 \times 3}$,

$$\begin{aligned} \Omega_{11} &= P^{-1}A + (P^{-1}A)^\top + P^{-1}BL + (P^{-1}BL)^\top - \tau_2 P^{-1}E + 2C^\top C + \frac{4}{\gamma_0} P^{-1}W(P^{-1}W)^\top \\ &\quad + P^{-1}\bar{E}A + (P^{-1}\bar{E}A)^\top + P^{-1}\bar{E}BL + (P^{-1}\bar{E}BL)^\top + \frac{4}{\gamma_0} P^{-1}\bar{E}W(P^{-1}\bar{E}W)^\top, \\ \Omega_{12} &= P^{-1}D + P^{-1}\bar{E}D + (P^{-1}\bar{E}A)^\top + (P^{-1}\bar{E}BL)^\top, \Omega_{13} = (P^{-1}A)^\top + (P^{-1}BL)^\top, \\ \Omega_{22} &= -\tau_2 P^{-1}E + 2H^\top H + \frac{4}{\gamma_0} P^{-1}W(P^{-1}W)^\top + P^{-1}\bar{E}D + (P^{-1}\bar{E}D)^\top + \frac{4}{\gamma_0} P^{-1}\bar{E}W(P^{-1}\bar{E}W)^\top, \\ \Omega_{23} &= [P^{-1}D]^\top, \Omega_{33} = -P^{-1} - (P^{-1})^\top + \frac{4}{\gamma_0} P^{-1}W(P^{-1}W)^\top. \end{aligned}$$

Moreover, note that

$$\Omega < 0 \Leftrightarrow \text{diag}[P, P, P] \Omega \text{diag}[P, P, P]^\top < 0 \Leftrightarrow \bar{\Omega} = [\bar{\Omega}_{ij}]_{3 \times 3} < 0, \quad (3.11)$$

where

$$\begin{aligned} \bar{\Omega}_{11} &= AP^\top + PA^\top + BY + (BY)^\top - \tau_2 EP^\top + 2PC^\top CP^\top + \frac{4}{\gamma_0} WW^\top, \\ &\quad + \bar{E}AP^\top + P(\bar{E}A)^\top + \bar{E}BY + (\bar{E}BY)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \\ \bar{\Omega}_{12} &= DP^\top + \bar{E}DP^\top + P(\bar{E}A)^\top + (\bar{E}BY)^\top, \bar{\Omega}_{13} = PA^\top + (BY)^\top, \\ \bar{\Omega}_{22} &= -\tau_2 EP^\top + 2PH^\top HP^\top + \frac{4}{\gamma_0} WW^\top + \bar{E}DP^\top + P(\bar{E}D)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \\ \bar{\Omega}_{23} &= PD^\top, \bar{\Omega}_{33} = -P^\top - P + \frac{4}{\gamma_0} WW^\top. \end{aligned}$$

We now use the Schur complement lemma and see that the condition (3.4) is equivalent to the condition $\bar{\Omega} < 0$, hence from (3.10), (3.11) it follows that

$$\begin{aligned} \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) - \tau_2 \mathbb{V}(x(t)) &\leq \tau_2 \mathbb{V}(x(t - \tau(t))) \\ &\quad - 2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2 + \gamma_0 \|\omega(t)\|^2. \end{aligned} \quad (3.12)$$

Taking (2.2), (3.12) and $-2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2 \leq 0$ into account, we obtain that

$$\mathcal{D}_C^\alpha(\mathbb{V}(x(t))) - \tau_2 \mathbb{V}(x(t)) \leq \tau_2 \mathbb{V}(x(t - \tau(t))) + \gamma_0 \|\omega(t)\|^2. \quad (3.13)$$

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Setting $\mathbb{M}(t) := \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) - \tau_2 \mathbb{V}(x(t))$ and using the Laplace transform and Lemma 2.2, we get from (3.13) that

$$\begin{aligned} \mathbb{V}(x(t)) &= \mathbb{V}(x(0))E_\alpha(\tau_2 t^\alpha) + \int_0^t \mathbb{M}(s)(t-s)^{\alpha-1} E_\alpha(\tau_2(t-s)^\alpha) ds \\ &\leq \mathbb{V}(x(0))E_\alpha(\tau_2 t^\alpha) + \int_0^t (t-s)^{\alpha-1} [\tau_2 V(x(s-\tau(s))) + \gamma_0 \|\omega(s)\|^2] E_\alpha(\tau_2(t-s)^\alpha) ds \\ &\leq \mathbb{V}(x(0))E_\alpha(\tau_2 t^\alpha) + (E_\alpha(\tau_2 t^\alpha) - 1) \sup_{-\tau_2 \leq s \leq t-\tau_1} \mathbb{V}(x(s)) + \gamma_0 E_\alpha(\tau_2 T^\alpha) \Gamma(\alpha) \mathcal{J}^\alpha \|\omega(t)\|^2. \end{aligned}$$

Since, by Lemma 2.1, the Mittag-Leffler function $E_\alpha(\cdot)$ is non-decreasing and $E_\alpha(\tau_2 T^\alpha) \geq 1$, we have

$$\begin{aligned} \sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s)) &\leq \sup_{-\tau_2 \leq s \leq 0} \mathbb{V}(x(s))E_\alpha(\tau_2 T^\alpha) + (E_\alpha(\tau_2 T^\alpha) - 1) \sup_{-\tau_2 \leq s \leq t-\tau_1} \mathbb{V}(x(s)) \\ &\quad + \gamma_0 E_\alpha(\tau_2 T^\alpha) \Gamma(\alpha) \sup_{s \in [0, T]} \mathcal{J}^\alpha \|\omega(s)\|^2. \end{aligned}$$

By induction and since the function $\sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s))$ is non-decreasing with respect to t , we obtain that

$$\sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s)) \leq \sup_{-\tau_2 \leq s \leq 0} \mathbb{V}(x(s)) \lambda_{\max}(P^{-1}E) + \beta_2^* \leq \beta_1 \|\varphi\|^2 + \beta_2^*, \quad t \in [0, T]. \quad (3.14)$$

where $\beta_2^* = \gamma_0 E_\alpha(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \Gamma(\alpha) \sup_{s \in [0, T]} \mathcal{J}^\alpha \|\omega(s)\|^2$ due to

$$\sup_{-\tau_2 \leq s \leq 0} \mathbb{V}(x(s)) \leq \lambda_{\max}(P^{-1}E) \|\varphi\|^2. \quad (3.15)$$

Consequently,

$$x(t)^\top P^{-1} E x(t) \leq \sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s)) \leq \beta_1 \|\varphi\|^2 + \beta_2^*, \quad t \in [0, T].$$

Besides, since

$$x(t)^\top P^{-1} E x(t) \geq \lambda_{\min}(P_1) \|x_1(t)\|^2, \quad t \in [0, T],$$

which shows that if $\|\varphi\|^2 \leq c_1$ then

$$\|x_1(t)\|^2 \leq \frac{\beta_1 c_1 + \beta_2^*}{\lambda_{\min}(P_1)}, \quad t \in [0, T]. \quad (3.16)$$

To get the estimation of $\|x(t)\|^2$, we need to estimate the second state $\|x_2(t)\|$. For this, we describe the second equation of (3.2) as

$$x_2(t) = -(\mathbb{A}_{22})^{-1} [\mathbb{A}_{21}x_1(t) + D_{21}x_1(t - \tau(t)) + D_{22}x_2(t - \tau(t)) + \omega_2(t)].$$

Taking the estimation (3.16) into account, we get

$$\begin{aligned}\|x_1(t - \tau(t))\|^2 &\leq c_1 + \frac{\beta_1 c_1 + \beta_2^*}{\lambda_{\min}(P_1)} = \beta_3 c_1 + \beta_4^*, \\ \|-(\mathbb{A}_{22})^{-1} \omega_2(t)\| &\leq \|(\mathbb{A}_{22})^{-1}\| \|W \omega(t)\| \leq \|(\mathbb{A}_{22})^{-1}\| \|W\| \sqrt{d} = \beta_7,\end{aligned}$$

where $\beta_4^* = \frac{\beta_2^*}{\lambda_{\min}(P_1)}$. Hence,

$$\|x_2(t)\| \leq \zeta^* + \beta \|x_2(t - \tau(t))\|, \quad t \in [0, T], \quad (3.17)$$

where $\zeta^* = 2\beta \sqrt{\beta_3 c_1 + \beta_4^*} + \beta_7$. On the other hand, using the inequality (3.17) for $t \in [0, \tau_1]$, we have $\|x_2(t)\| \leq \zeta^* + \beta \sqrt{c_1}$, because of $t - \tau_2 \leq t - \tau(t) \leq t - \tau_1 \leq 0$, and $\|x_2(t - \tau(t))\|^2 \leq c_1$. By induction, for $t \in [i\tau_1, (i+1)\tau_1] \cap [0, T]$, $i\tau_1 \leq T$, we get $\|x_2(t)\| \leq \zeta^* \sum_{k=0}^i \beta^k + \beta^{i+1} \sqrt{c_1}$, and hence

$$\|x_2(t)\| \leq \zeta^* \beta_5 + \beta_6 \sqrt{c_1}, \quad t \in [0, T]. \quad (3.18)$$

Finally, combining conditions (3.16), (3.18) with (3.5) and noting $\beta_2^* \leq \beta_2$, $\zeta^* \leq \zeta$, we obtain that

$$\begin{aligned}\|x(t)\|^2 &= \|x_1(t)\|^2 + \|x_2(t)\|^2 \leq \frac{\beta_2^* c_1 + \beta_3}{\lambda_{\min}(P_1)} + (\zeta^* \beta_5 + \beta_6 \sqrt{c_1})^2 \\ &\leq \frac{\beta_2 c_1 + \beta_3}{\lambda_{\min}(P_1)} + (\zeta \beta_5 + \beta_6 \sqrt{c_1})^2 \leq c_2, \quad t \in [0, T].\end{aligned}$$

Step 3. The γ -optimal level condition. From (3.12) and (3.14) it follows that

$$\begin{aligned}\|z(t)\|^2 &\leq 2\|Cx(t)\|^2 + 2\|Hx(t - \tau(t))\|^2 \\ &\leq \gamma_0 \|\omega(t)\|^2 + 2\tau_2(\beta_1 \|\varphi\|^2 + \beta_2^*) - \mathcal{D}_C^\alpha(\mathbb{V}(x(t))).\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathcal{J}^\alpha \|z(t)\|^2 &\leq \gamma_0 I^\alpha \|\omega(t)\|^2 + \mathcal{J}^\alpha (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) - \mathcal{J}^\alpha \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) \\ &= \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{t^\alpha}{\Gamma(\alpha+1)} + [-\mathbb{V}(x(t)) + \mathbb{V}(x(0))] \\ &\leq \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{t^\alpha}{\Gamma(\alpha+1)} + \mathbb{V}(x(0)) \\ &\leq \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{t^\alpha}{\Gamma(\alpha+1)} + \mathbb{V}(x(0)) \\ &\leq \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{T^\alpha}{\Gamma(\alpha+1)} + \lambda_{\max}(P^{-1}E) \|\varphi\|^2 \\ &= \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + \left(2\tau_2 \beta_1 \frac{T^\alpha}{\Gamma(\alpha+1)} + \lambda_{\max}(P^{-1}E)\right) \|\varphi\|^2 + 2\tau_2 \beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &= \gamma_0 (\mathcal{J}^\alpha \|\omega(t)\|^2 + c_0 \|\varphi\|^2) + 2\tau_2 \beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

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where $c_0 = \frac{(2\tau_2\beta_1 \frac{T^\alpha}{\Gamma(\alpha+1)} + \lambda_{\max}(P^{-1}E))}{\gamma_0}$. Moreover, for all $t \in [0, T]$ we have

$$\sup_{t \in [0, T]} \mathcal{J}^\alpha \|z(t)\|^2 \leq \gamma_0 \left(\sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2 + c_0 \|\varphi\|^2 \right) + 2\tau_2\beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)},$$

which leads to

$$\begin{aligned} \frac{\sup_{t \in [0, T]} \mathcal{J}^\alpha \|z(t)\|^2}{c_0 \|\varphi\|^2 + \sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} &\leq \gamma_0 + \frac{2\tau_2\beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)}}{c_0 \|\varphi\|^2 + \sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} \\ &\leq \gamma_0 + \frac{2\tau_2\beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)}}{\sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} \\ &= \gamma_0 + 2\tau_2\gamma_0 E_{\alpha, \alpha}(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \Gamma(\alpha) \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &= \gamma_0 \left(1 + 2\tau_2 E_{\alpha, \alpha}(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \frac{T^\alpha}{\alpha} \right) = \gamma. \end{aligned}$$

Therefore, we finally obtain that

$$\frac{\|z(t)\|_\alpha^2}{c_0 \|\varphi\|^2 + \|\omega(t)\|_\alpha^2} = \frac{\sup_{t \in [0, T]} \mathcal{J}^\alpha \|z(t)\|^2}{c_0 \|\varphi\|^2 + \sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} \leq \gamma,$$

which completes the proof of the theorem. $\square$

REMARK 3.1 In Theorem 3.1, we assume that c_1, c_2, γ, d are given positive parameters. Therefore, to check the conditions of the theorem, we have to prescribe these parameters firstly. Since the matrix relation (3.3) is not a linear matrix inequality, but we can reduce it into a single strict LMI by some matrix transformation. Also, since the scalars c_1, c_2 are not involved in the LMI condition (3.4) we first find the unknowns of LMI (3.4) by using LMI Tollbox algorithm (Gahinet et al., 1985) and then verify the inequality (3.5).

REMARK 3.2 It is noted that in the fractional-order system context, the H_∞ finite-time controller are designed in (Zhang et al., 2014; Mathiyalagan and Balachandran, 2016; Liu et al., 2018, Osorio-Gorllio et al., 2018, Thuan et al., 2020) for the systems with or without delays. In this paper, the design method is proposed for the singular FDEs with nondifferentiable time-varying delay, which can not be applicable in the mentioned papers. Theorem 3.1 provides an analytical approach to the design of finite-time H_∞ controllers for singular FDEs time-delay systems and the obtained result can be considered as extensions of those in the mentioned papers to time-varying delay systems.

EXAMPLE 3.2 Consider the singular FDEs (2.1), where $\alpha = 0.1, d = 2$ and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, D = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$W = 10^{-15} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, C = 10^{-15} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, H = 10^{-15} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\tau(t) = 0.1 + 0.1|\sin t|,$$

We have $d = 2, \tau_1 = 0.1$. For $\gamma = 0.5, c_1 = 0.5, c_2 = 4, T = 20$, the solutions of LMI (3.4) can be determined as

$$P = \begin{pmatrix} 1979.3 & 0 \\ 0 & 112.4 \end{pmatrix}, \quad Y = \begin{pmatrix} 631.4239 & 11.2376 \\ 11.2376 & 11.2376 \end{pmatrix}.$$

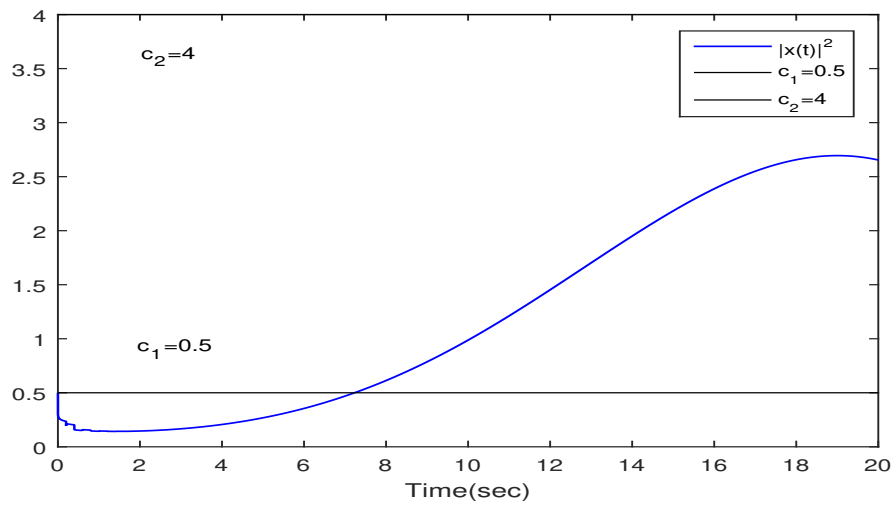


FIG. 1. The time history of $\|x(t)\|^2$

Hence,

$$L = Y(P^{-1})^\top = \begin{bmatrix} 0.3190 & 0.1000 \\ 0.0057 & 0.1000 \end{bmatrix}, \quad \mathbb{A} = A + BL = \begin{bmatrix} -0.4133 & -0.0000 \\ -0.0057 & -0.2000 \end{bmatrix}, \quad \mathbb{A}_{22} = -0.2000.$$

Moreover, we have

$$\gamma_0 = 0.1707, \beta = 0.5, \beta_1 = 0.0012, \beta_2 = 1.6462, \beta_3 = 1,$$

$$\beta_4 = 0.0008, \beta_5 = 2, \beta_6 = 0.5, \beta_7 = 1.4142e - 14, \xi = 0.7077,$$

for verifying the condition (3.5):

$$\left(\frac{\beta_2 c_1 + \beta_3}{\lambda_{\min}(P_1)} + [\zeta \beta_5 + \beta_6 \sqrt{c_1}]^2 \right) = 3.1301 \leq c_2 = 4.$$

Then, by Theorem 3.1 the finite-time H_∞ control of the system is solvable with the following state feedback controller

$$u(t) = \begin{pmatrix} 0.3190 & 0.1000 \\ 0.0057 & 0.1000 \end{pmatrix} x(t), \quad t \in [0, 20].$$

Figure 1 and Figure 2 show the time history of $\|x(t)\|^2$ and the state response of the system with the initial condition $\varphi(t) = [0.5, 0.5]^T$, respectively.

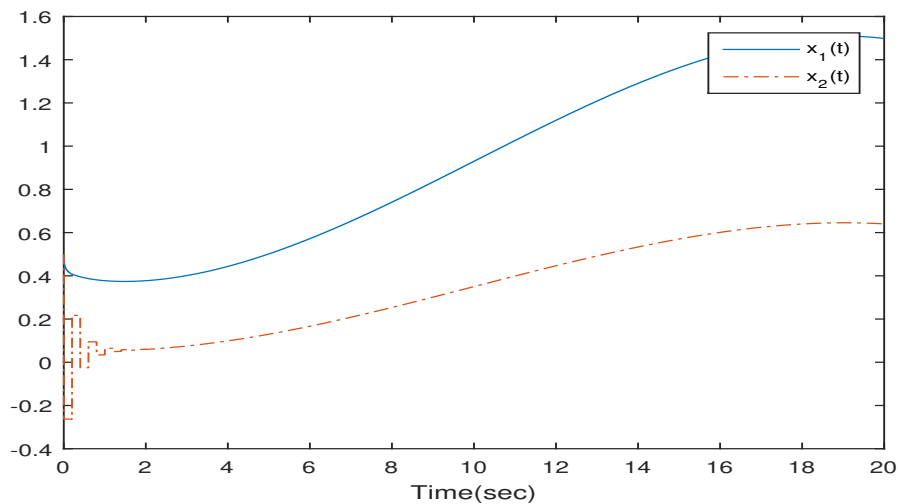


FIG. 2. The state response of the solutions

4. Conclusion

In summary, we have studied finite-time H_∞ control of linear singular FDEs with interval time-varying delay. Based on the fractional calculus and singular value decomposition approach, a set of new sufficient conditions are established for solving the problem that provides the designing the state feedback controllers and guaranteeing the resulting closed-loop system to be regular, impulse-free and finite-time stable with a given H_∞ performance index. The obtained results can be considered as extensions of those in various existing papers to time-varying delay systems.

Acknowledgments

The research of the first author is supported by the Chiang Mai University, Thailand. The research of the second and third authors is supported by National Foundation for Science and Technology Development, Vietnam (NAFOSTED 101.01-2021.01). The authors are grateful to anonymous reviewers for their valuable comments and suggestions, which allow us to improve the paper.

Conflict of interest

The authors declare that there is no conflict of interest.

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doi:10.1093/imamci/dri000

Finite-time H_∞ control of linear singular fractional differential equations with time-varying delay

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[Received]

In this paper, we propose an efficient analytical approach based on fractional calculus and singularity value theory to designing the finite-time H_∞ controller for linear singular fractional differential equations with time-varying delay. By introducing new fractional-order H_∞ norm, the state feedback controller is designed to guarantee that the closed-loop system is singular, impulse-free and finite-time stable with prescribed H_∞ performance. New sufficient conditions for designing the H_∞ finite-time controller are presented. The results of this paper improve the corresponding ones of integer-order singular systems with time-varying delay. Finally, a numerical example demonstrates the validity and effectiveness of the proposed theoretical results.

Keywords: Fractional derivative, Finite-time control, Singular systems, Mittag-Leffler functions, Laplace transform, time-varying delays.

1. Introduction

During the past decades, the research on fractional differential equations (FDEs) has attracted increasing interest and has numerous applications in science and engineering. There has been a significant theoretical development in FDEs (Podlubny, 1999; Kilbas et al., 2006; Lakshmikantham and Vatsala, 2007; Kamal et al., 2015; Buslowicz, 2016; Wu and Baleanu, 2018). So far, there has been a growing research interest on the H_∞ control problem of dynamical systems (Francis, 1987; Petersen et al., 2000). The research activities for the problem have been developed for integer-order systems (Fridman and Shaked, 2003; Haidi and Mohamad, 2020; Zhang et al., 2018). For fractional-order systems, there are some results about H_∞ control for FDEs. In (Boukal et al., 2017) the authors studied H_∞ observer-based control of FDEs with gain parametrization. The authors of (Shen and Lam, 2014) designed state feedback H_∞ controllers of commensurate FDEs, meanwhile the paper (N'Doye et al, 2015) designed output feedback H_∞ controllers of Glucose-insulin FDEs without delays.

In practice, one concerns the boundedness of solutions during a prescribed time interval, which is called finite-time stability (FTS) (Donato, 1961; Amato et al., 2014). FTS involves dynamical systems whose solutions are bounded over finite time. Many significant results on the finite-time stability and control have been obtained in the last decade. For integer-order systems, paper (Guo et al. 2013) proposed significant criteria for designing finite-time H_∞ controllers of linear systems without delay by using Lyapunov function method and linear matrix inequality techniques. In (Zong et al., 2015, Sun and Zhao, 2021), the authors studied finite-time H_∞ control for switched nonlinear discrete systems with delays by applying Lyapunov-Krasovskii method. For fractional-order systems, the authors of (Liu et al., 2018, Thuan et al., 2020) investigated finite-time H_∞ control for FDEs without delays. On the other

hand, the singular (or differential-algebraic, implicit, descriptor) systems have received substantial interest from researchers due to their widespread practical applications in control engineering (Dai, 1989; Wu et al. 2013, Fridman, 2002). When studying stability and control of singular systems, an aspect to be considered is ensuring the admissibility (stability, impulse-free and regularity) of the system. By employing the singular value decomposition and Lyapunov function method, there are various significant results on the stability and control problem proposed in (Mahmoud, 2009; Osorio-Gordillo, 2018; Phat et al., 2020, Chen et al., 2021; Du et al., 2021) for integer-order singular systems with delays. For singular FDEs with delays, the authors of (Pang and Jiang, 2014, Zhang et al., 2014, Liu et al., 2017, Thanh et al., 2020) proposed delay-dependent conditions for the asymptotic and finite-time stability. It is worth noting that there are no results on finite-time H_∞ control of singular FDEs with delays, since the classical solutions and methodology of singular FDEs cannot be applied for these systems.

In this paper, we study the problem of finite-time H_∞ control for linear singular FDEs with interval time-varying delay. The contributions of our paper are summarized as follows.

- (i) The singular FDEs under consideration deals with interval nondifferentiable time-varying delay and normed bounded perturbations.
- (ii) A novel approach based on the fractional calculus (Caputo derivative, Laplace transform, the Mittag-Leffler function) and the singularity value theory is proposed to solve the finite-time H_∞ control problem.
- (iii) By introducing a new fractional-order H_∞ norm, the delay-dependent conditions are derived and presented in the form of tractable matrix inequalities.
- (iv) Comparing with the existing works, the proposed results can be applied to more general FDEs with delays including the works of (Zhang et al., 2014, Shen & Lam, 2014, Mathiyalgan and Balakchandran, 2016, Liu et al., 2018, Osorio-Gordillo et al., 2018, Thuan et al., 2020).

The paper is organized as follows. Section 2 introduces basic definitions of Caputo derivative with its properties and the control problem to be considered. Main result on the design of finite-time H_∞ controllers is presented in Section 3 with a numerical example and simulations. Finally, Section 4 summarizes the conclusions.

2. Preliminaries

The following notations are used throughout this paper. $\mathbb{C}$ denotes the complex spaces; $\mathbb{R}^n$ denotes the n -dimensional Euclidean space; $\mathbb{R}^{n \times m}$ stands for the space of all $(n \times m)$ -matrices; $\lambda(A)$ denotes the set of all eigenvalues of A ; $\lambda_{\min}(A) = \min\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$; A is positive definite ($A > 0$) if $(Ay, y) > 0$ for all $y \neq 0$; $L^1[0, b]$ denotes the space of integrable functions on $[0, b]$; ΔV denotes the gradient of vector function $V(\cdot)$, $[a]$ denotes the integer part of a .

In order to describe the problem, we will recall some basic definitions about fractional calculus and several useful lemmas of (Podlubny, 1999; Kilbas et al., 2006). For $\alpha \in (0, 1)$ and $x \in L^1[0, T]$, the Riemann-Liouville integral and the Riemann-Liouville derivative are respectively defined by

$$\begin{aligned} \mathcal{I}^\alpha x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \\ \mathcal{D}_R^\alpha x(t) &= \frac{d}{dt} (\mathcal{I}^{1-\alpha} x(t)), \end{aligned}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$, $\operatorname{Re}(s) > 0$ is the gamma function. The Caputo derivative is then defined as

$$\mathcal{D}_C^\alpha x(t) = \mathcal{D}_R^\alpha (x(t) - x(0)).$$

Recall the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and the Laplace transform $\mathcal{L}[x(t)](s)$ defined respectively by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \mathcal{L}[x(t)](s) = \int_0^{\infty} e^{-st} x(t) dt.$$

For $\beta = 1$, denote $E_\alpha(z) := E_{\alpha,1}(z)$.

LEMMA 2.1 (Kilbas et al., 2006). For $\alpha > 0, z \geq 0$, the Mittag-Leffler function $E_\alpha(z)$ is an increasing function and $E_\alpha(z) \geq 1$.

LEMMA 2.2 (Kilbas et al., 2006) Let $x, y \in L^1[0, T], \alpha \in (0, 1), h > 0$, we have

- (a) $\mathcal{L}[\mathcal{D}_C^\alpha x(t)](\xi) = \xi^\alpha \mathcal{L}[x(t)](\xi) - \xi^{\alpha-1} x(0)$.
- (b) $\mathcal{L}[t^{\alpha h} E_\alpha^{(k)}(ht^\alpha)](\xi) = \frac{k! \xi^{\alpha-1}}{(\xi^\alpha - h)^{k+1}}, k = 1, 2, \dots, \operatorname{Re}(\xi) > h^{1/\alpha}$.
- (c) $\mathcal{L}[x * y(t)](\xi) = \mathcal{L}[x(t)](\xi) \cdot \mathcal{L}[y(t)](\xi)$,

where the convolution of $x(t)$ and $y(t)$ is defined by $x * y(t) = \int_0^t x(t - \tau) y(\tau) d\tau$.

LEMMA 2.3 (Trinh & Tuan, 2018) Let $x(t)$ be a continuous function, $V(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a convex and differentiable function such that $V(0) = 0$. We have

$$\mathcal{D}_C^\alpha V(x(t)) \leq \langle \Delta V(x(t)), \mathcal{D}_C^\alpha x(t) \rangle, \quad t \geq 0.$$

Consider the following singular FDEs with time-varying delay:

$$\begin{cases} E \mathcal{D}_C^\alpha x(t) &= Ax(t) + Dx(t - \tau(t)) + Bu(t) + W\omega(t), \\ z(t) &= Cx(t) + Hx(t - \tau(t)), \\ x(s) &= \varphi(s), s \in [-\tau_2, 0], \end{cases} \quad (2.1)$$

where $0 < \alpha < 1; x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $\omega(t) \in \mathbb{R}^p$ is the disturbance, E is a singular matrix: $\operatorname{rank} E = r < n; A, D, C, H \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, W \in \mathbb{R}^{n \times p}$ are given constant matrices, $\varphi(\cdot)$ is the continuous initial function, the delay function $\tau(t)$ satisfies

$$\exists \tau_1, \tau_2 > 0: \quad 0 < \tau_1 \leq \tau(t) \leq \tau_2, \quad t \geq 0;$$

the disturbance $\omega(t)$ satisfies

$$\exists d > 0: \quad \sup_{t \geq 0} \|\omega(t)\|^2 \leq d, \quad t \geq 0. \quad (2.2)$$

As in (Dai, 1989) we recall that system (2.1) is regular if $\det(sE - A)$ is not identically zero and is impulse-free if $\det(\det(sE - A)) = \operatorname{rank} E$, for some $c \in \mathbb{C}$.

DEFINITION 2.1 For scalars $c_1 > 0, c_2 > 0, T > 0$, system (2.1) is FTS w.r.t. (c_1, c_2, T) if it is regular, impulse-free and the following relation holds:

$$\sup_{s \in [-\tau_2, 0]} \|\varphi(s)\|^2 \leq c_1 \implies \|x(t)\|^2 \leq c_2, \quad t \in [0, T].$$

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Let us introduce the fractional-order norm of $f(t)$ defined by

$$\|f\|_\alpha := \left\{ \sup_{s \in [0, T]} \mathcal{J}^\alpha \|f(s)\|^2 \right\}^{1/2}.$$

It is notable that $\|f\|_\alpha$ defines a norm in $L^1[0, T]$, and for $\alpha = 1$ this norm is reduced to the L_2 norm of $f(t)$ because of

$$\sup_{s \in [0, T]} \mathcal{J}^\alpha \|f(s)\|^2 = \sup_{s \in [0, T]} \int_0^s \|f(\tau)\|^2 d\tau = \int_0^T \|f(\tau)\|^2 d\tau.$$

DEFINITION 2.2 For scalars $c_1 > 0, c_2 > 0, T > 0, \gamma > 0$, the finite-time H_∞ control problem for system (2.1) is solvable if there exist a number $c_0 > 0$ and a control $u(t) = Lx(t)$ such that the resulting closed-loop system is FTS w.r.t. (c_1, c_2, T) and

$$\sup \frac{\|z(s)\|_\alpha^2}{c_0 \|\varphi\|^2 + \|\omega(s)\|_\alpha^2} \leq \gamma, \quad (2.3)$$

where the supremum is taken over all non-zero initial conditions $\varphi(\cdot)$ and disturbances $\omega(t)$ satisfying (2.2).

3. Finite-time H_∞ control design

This section presents delay-dependent criteria for designing the finite-time H_∞ controllers. Consider singular system (2.1), where the singular matrix E is, without of generality as in (Fridman, 2002), assumed that $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. By using the feedback controller $u(t) = Lx(t)$, where the matrix $L \in R^{m \times n}$ will be designed, the closed-loop system of (2.1) has the form

$$\begin{cases} E \mathcal{D}_C^\alpha x(t) &= \mathbb{A}x(t) + Dx(t - \tau(t)) + W\omega(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau_2, 0], \end{cases} \quad (3.1)$$

where $\mathbb{A} = A + BL$. Setting

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix}; \quad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}; \quad W\omega(t) = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix},$$

under the state transformation $x(t) = [x_1(t), x_2(t)]$, $x_1(t) \in R^r$, $x_2(t) \in R^{n-r}$, the closed loop system (3.1) has the form of fractional differential-algebraic equations

$$\begin{cases} \mathcal{D}_C^\alpha x_1(t) &= \mathbb{A}_{11}x_1(t) + \mathbb{A}_{12}x_2(t) + D_{11}x_1(t - \tau(t)) + D_{12}x_2(t - \tau(t)) + \omega_1(t), \\ 0 &= \mathbb{A}_{21}x_1(t) + \mathbb{A}_{22}x_2(t) + D_{21}x_1(t - \tau(t)) + D_{22}x_2(t - \tau(t)) + \omega_2(t), \\ x(t) &= \varphi(t), \quad t \in [-\tau_2, 0]. \end{cases} \quad (3.2)$$

For given matrices E, A, B, W and matrices A_{22}^{-1}, Y , non-singular matrix $P \in R^{n \times n}$ to be defined, we introduce the following notations.

$$\bar{E} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \quad \gamma_0 = \frac{\gamma}{1 + 2\tau_2 E_\alpha(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \frac{T^\alpha}{\alpha}},$$

$$\begin{aligned}
\beta &= \max\{\|\mathbb{A}_{22}^{-1}\mathbb{A}_{21}\|, \|\mathbb{A}_{22}^{-1}D_{21}\|, \|\mathbb{A}_{22}^{-1}D_{22}\|\}, \\
\beta_1 &= \lambda_{\max}(P^{-1}E) \sum_{j=0}^{[T/\tau_1]+1} (E_\alpha(\tau_2 T^\alpha) - 1)^j E_\alpha(\tau_2 T^\alpha), \\
\beta_2 &= \gamma_0 E_\alpha(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]+1} (E_\alpha(\tau_2 T^\alpha) - 1)^j \frac{T^\alpha}{\alpha} d, \\
\beta_3 &= 1 + \frac{\beta_1}{\lambda_{\min}(P_1)}, \beta_4 = \frac{\beta_2}{\lambda_{\min}(P_1)}, \beta_5 = \max_{i=0,1,2,\dots,[T/\tau_1]} \sum_{k=0}^i \beta^k, \beta_6 = \max_{i=0,1,2,\dots,[T/\tau_1]} \beta^{i+1}, \\
\beta_7 &= \|\mathbb{A}_{22}^{-1}\| \|W\| \sqrt{d}, \quad \zeta = 2\beta \sqrt{\beta_3 c_1 + \beta_4} + \beta_7, \\
\mathbb{E}_{11} &= AP^\top + PA^\top + BY + (BY)^\top - \tau_2 EP^\top + \frac{4}{\gamma_0} WW^\top + \bar{E}AP^\top + P(\bar{E}A)^\top \\
&\quad + \bar{E}BY + (\bar{E}BY)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \\
\mathbb{E}_{12} &= DP^\top + \bar{E}DP^\top + P(\bar{E}A)^\top + (\bar{E}BY)^\top, \mathbb{E}_{13} = PA^\top + (BY)^\top, \\
\mathbb{E}_{22} &= -\tau_2 EP^\top + \frac{4}{\gamma_0} WW^\top + \bar{E}DP^\top + P(\bar{E}D)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \mathbb{E}_{33} = -P^\top - P + \frac{4}{\gamma_0} WW^\top.
\end{aligned}$$

THEOREM 3.1 For scalars $\gamma > 0, c_1 > 0, c_2 > 0, T > 0$, problem of finite-time H_∞ control for system (2.1) is solvable if there exist a non-singular matrix P and a matrix Y such that the following conditions hold:

$$EP^\top = PE^\top \geq 0, \quad (3.3)$$

$$\begin{pmatrix}
\mathbb{E}_{11} & \mathbb{E}_{12} & \mathbb{E}_{13} & PC^\top & 0 \\
* & \mathbb{E}_{22} & PD^\top & 0 & PH^\top \\
* & * & \mathbb{E}_{33} & 0 & 0 \\
* & * & * & -0.5I & 0 \\
* & * & * & * & -0.5I
\end{pmatrix} < 0, \quad (3.4)$$

$$\frac{\beta_2 c_1 + \beta_3}{\lambda_{\min}(P_1)} + (\zeta \beta_5 + \beta_6 \sqrt{c_1})^2 \leq c_2. \quad (3.5)$$

The state feedback control is defined by $u(t) = Y(P^{-1})^\top x(t)$, $t \in [0, T]$.

Proof.

Step 1. Regularity and impulse-free. Setting $P^{-1} = \begin{pmatrix} P_1 & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ and using condition (3.3), we have

$$\begin{aligned}
P^{-1}E &= P^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ P_{21} & 0 \end{pmatrix} \geq 0, \\
E^\top [P^{-1}]^\top &= \begin{pmatrix} P_1^\top & P_{21}^\top \\ 0 & 0 \end{pmatrix} \geq 0,
\end{aligned}$$

and hence

$$P_{21} = 0, P_1 = P_1^\top \geq 0, P^{-1}E = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, P^{-1} = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix}. \quad (3.6)$$

Since matrix P is nonsingular and $P^{-1} = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix}$, from (3.6) it follows that $\det(P_1) \neq 0, P_1 > 0$.

Moreover, by the Schur complement lemma, the LMI condition (3.4) gives

$$AP^\top + PA^\top + BY + (BY)^\top - \tau_2 EP^\top + \bar{E}AP^\top + P(\bar{E}A)^\top + \bar{E}BY + (\bar{E}BY)^\top < 0,$$

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and hence

$$\begin{aligned} & P^{-1}(A + BY(P^{-1})^\top) + [A + BY(P^{-1})^\top]^\top (P^{-1})^\top - \tau_2 P^{-1}E \\ & + P^{-1}\bar{E}A + (\bar{E}A)^\top (P^{-1})^\top + P^{-1}\bar{E}BY(P^{-1})^\top + [P^{-1}\bar{E}BY(P^{-1})^\top]^\top < 0. \end{aligned}$$

Letting $L = Y(P^{-1})^\top$ gives

$$P^{-1}\mathbb{A} + \mathbb{A}^\top (P^{-1})^\top - \tau_2 P^{-1}E + P^{-1}\bar{E}\mathbb{A} + (P^{-1}\bar{E}\mathbb{A})^\top < 0. \quad (3.7)$$

Besides, we describe the matrices $P^{-1}\mathbb{A}$, $P^{-1}E$, $P^{-1}\bar{E}\mathbb{A}$ as follows

$$\begin{aligned} P^{-1}\mathbb{A} &= \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix} = \begin{pmatrix} P_1\mathbb{A}_{11} + P_{12}\mathbb{A}_{21} & P_1\mathbb{A}_{12} + P_{12}\mathbb{A}_{22} \\ P_{22}\mathbb{A}_{21} & P_{22}\mathbb{A}_{22} \end{pmatrix}, \\ P^{-1}E &= \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, \\ P^{-1}\bar{E}\mathbb{A} &= P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \mathbb{A} = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix} = \begin{pmatrix} P_{12}\mathbb{A}_{21} & P_{12}\mathbb{A}_{22} \\ P_{22}\mathbb{A}_{21} & P_{22}\mathbb{A}_{22} \end{pmatrix}. \end{aligned}$$

Then, combining with the derived inequality (3.7) we obtain that

$$(P_{22})\mathbb{A}_{22} + \mathbb{A}_{22}^\top (P_{22})^\top < 0,$$

which gives $\det(\mathbb{A}_{22}) \neq 0$ and then the system is regular and impulse-free.

Step 2. Finite-time stability. Consider the functional $\mathbb{V}(x(t)) = x(t)^\top P^{-1}Ex(t)$. Since $\mathbb{V}(\cdot)$ is a non-negative convex function satisfying condition of Lemma 2.3, we then can estimate the Caputo derivative of $\mathbb{V}(x(t))$ as

$$\mathcal{D}_C^\alpha(\mathbb{V}(x(t))) \leq 2x(t)^\top P^{-1}E\mathcal{D}_C^\alpha(x(t)), \quad t \geq 0.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) &\leq 2x(t)^\top P^{-1} \left(Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t) \right) \\ &\leq 2x(t)^\top P^{-1} \left(Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t) \right) \\ &\quad - \tau_2 x(t - \tau(t))^\top P^{-1}Ex(t - \tau(t)) + \tau_2 V(x(t - \tau(t))) \\ &\quad - \tau_2 x(t)^\top P^{-1}Ex(t) + \tau_2 V(x(t)) + 2\|Cx(t)\|^2 \\ &\quad + 2\|Hx(t - \tau(t))\|^2 - 2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2. \end{aligned} \quad (3.8)$$

Multiplying left both sides of (2.1) by $2(E\mathcal{D}_C^\alpha x(t))^\top P^{-1}$, $2x(t)^\top P^{-1}\bar{E}$, $2x(t - \tau(t))^\top P^{-1}\bar{E}$, we get

$$\begin{aligned} 0 &= -2(E\mathcal{D}_C^\alpha x(t))^\top P^{-1}E\mathcal{D}_C^\alpha x(t) + 2(E\mathcal{D}_C^\alpha x(t))^\top P^{-1}[Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t)], \\ 0 &= -2x(t)^\top P^{-1}\bar{E}E\mathcal{D}_C^\alpha x(t) + 2x(t)^\top P^{-1}\bar{E}[Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t)], \\ 0 &= -2x(t - \tau(t))^\top P^{-1}\bar{E}E\mathcal{D}_C^\alpha x(t) + 2x(t - \tau(t))^\top P^{-1}\bar{E}[Ax(t) + Dx(t - \tau(t)) + BLx(t) + W\omega(t)]. \end{aligned} \quad (3.9)$$

Using the following inequalities

$$\begin{aligned} \frac{4}{\gamma_0} x^\top P^{-1} \bar{E} W (P^{-1} \bar{E} W)^\top x + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2x^\top P^{-1} \bar{E} W \omega(t), \\ \frac{4}{\gamma_0} x(t)^\top P^{-1} W (P^{-1} W)^\top x(t) + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2x(t)^\top P^{-1} W \omega(t), \\ \frac{4}{\gamma_0} (E \mathcal{D}_C^\alpha x(t))^\top P^{-1} W (P^{-1} W)^\top (E \mathcal{D}_C^\alpha x(t)) + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2(E \mathcal{D}_C^\alpha x(t))^\top P^{-1} W \omega(t), \\ \frac{4}{\gamma_0} x(t - \tau(t))^\top P^{-1} \bar{E} W (P^{-1} \bar{E} W)^\top x(t - \tau(t)) + \frac{\gamma_0}{4} \|\omega(t)\|^2 &\geq 2x(t - \tau(t))^\top P^{-1} \bar{E} W \omega(t), \end{aligned}$$

and from (3.8)-(3.9) it follows that

$$\begin{aligned} \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) &\leq \zeta(t)^\top \Omega \zeta(t) + \tau_2 \mathbb{V}(x(t - \tau(t))) + \tau_2 \mathbb{V}(x(t)) \\ &\quad - 2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2 + \gamma_0 \|\omega(t)\|^2, \end{aligned} \quad (3.10)$$

where $\zeta(t) = [x(t), x(t - \tau(t)), E \mathcal{D}_C^\alpha x(t)]$, $\Omega = [\Omega_{ij}]_{3 \times 3}$,

$$\begin{aligned} \Omega_{11} &= P^{-1}A + (P^{-1}A)^\top + P^{-1}BL + (P^{-1}BL)^\top - \tau_2 P^{-1}E + 2C^\top C + \frac{4}{\gamma_0} P^{-1}W(P^{-1}W)^\top \\ &\quad + P^{-1}\bar{E}A + (P^{-1}\bar{E}A)^\top + P^{-1}\bar{E}BL + (P^{-1}\bar{E}BL)^\top + \frac{4}{\gamma_0} P^{-1}\bar{E}W(P^{-1}\bar{E}W)^\top, \\ \Omega_{12} &= P^{-1}D + P^{-1}\bar{E}D + (P^{-1}\bar{E}A)^\top + (P^{-1}\bar{E}BL)^\top, \quad \Omega_{13} = (P^{-1}A)^\top + (P^{-1}BL)^\top, \\ \Omega_{22} &= -\tau_2 P^{-1}E + 2H^\top H + \frac{4}{\gamma_0} P^{-1}W(P^{-1}W)^\top + P^{-1}\bar{E}D + (P^{-1}\bar{E}D)^\top + \frac{4}{\gamma_0} P^{-1}\bar{E}W(P^{-1}\bar{E}W)^\top, \\ \Omega_{23} &= [P^{-1}D]^\top, \quad \Omega_{33} = -P^{-1} - (P^{-1})^\top + \frac{4}{\gamma_0} P^{-1}W(P^{-1}W)^\top. \end{aligned}$$

Moreover, note that

$$\Omega < 0 \Leftrightarrow \text{diag}[P, P, P] \Omega \text{diag}[P, P, P]^\top < 0 \Leftrightarrow \bar{\Omega} = [\bar{\Omega}_{ij}]_{3 \times 3} < 0, \quad (3.11)$$

where

$$\begin{aligned} \bar{\Omega}_{11} &= AP^\top + PA^\top + BY + (BY)^\top - \tau_2 EP^\top + 2PC^\top CP^\top + \frac{4}{\gamma_0} WW^\top, \\ &\quad + \bar{E}AP^\top + P(\bar{E}A)^\top + \bar{E}BY + (\bar{E}BY)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \\ \bar{\Omega}_{12} &= DP^\top + \bar{E}DP^\top + P(\bar{E}A)^\top + (\bar{E}BY)^\top, \quad \bar{\Omega}_{13} = PA^\top + (BY)^\top, \\ \bar{\Omega}_{22} &= -\tau_2 EP^\top + 2PH^\top HP^\top + \frac{4}{\gamma_0} WW^\top + \bar{E}DP^\top + P(\bar{E}D)^\top + \frac{4}{\gamma_0} \bar{E}W(\bar{E}W)^\top, \\ \bar{\Omega}_{23} &= PD^\top, \quad \bar{\Omega}_{33} = -P^\top - P + \frac{4}{\gamma_0} WW^\top. \end{aligned}$$

We now use the Schur complement lemma and see that the condition (3.4) is equivalent to the condition $\bar{\Omega} < 0$, hence from (3.10), (3.11) it follows that

$$\begin{aligned} \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) - \tau_2 \mathbb{V}(x(t)) &\leq \tau_2 \mathbb{V}(x(t - \tau(t))) \\ &\quad - 2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2 + \gamma_0 \|\omega(t)\|^2. \end{aligned} \quad (3.12)$$

Taking (2.2), (3.12) and $-2\|Cx(t)\|^2 - 2\|Hx(t - \tau(t))\|^2 \leq 0$ into account, we obtain that

$$\mathcal{D}_C^\alpha(\mathbb{V}(x(t))) - \tau_2 \mathbb{V}(x(t)) \leq \tau_2 \mathbb{V}(x(t - \tau(t))) + \gamma_0 \|\omega(t)\|^2. \quad (3.13)$$

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Setting $\mathbb{M}(t) := \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) - \tau_2 \mathbb{V}(x(t))$ and using the Laplace transform and Lemma 2.2, we get from (3.13) that

$$\begin{aligned} \mathbb{V}(x(t)) &= \mathbb{V}(x(0))E_\alpha(\tau_2 t^\alpha) + \int_0^t \mathbb{M}(s)(t-s)^{\alpha-1} E_\alpha(\tau_2(t-s)^\alpha) ds \\ &\leq \mathbb{V}(x(0))E_\alpha(\tau_2 t^\alpha) + \int_0^t (t-s)^{\alpha-1} [\tau_2 V(x(s-\tau(s))) + \gamma_0 \|\omega(s)\|^2] E_\alpha(\tau_2(t-s)^\alpha) ds \\ &\leq \mathbb{V}(x(0))E_\alpha(\tau_2 t^\alpha) + (E_\alpha(\tau_2 T^\alpha) - 1) \sup_{-\tau_2 \leq s \leq t-\tau_1} \mathbb{V}(x(s)) + \gamma_0 E_\alpha(\tau_2 T^\alpha) \Gamma(\alpha) \mathcal{J}^\alpha \|\omega(t)\|^2. \end{aligned}$$

Since, by Lemma 2.1, the Mittag-Leffler function $E_\alpha(\cdot)$ is non-decreasing and $E_\alpha(\tau_2 T^\alpha) \geq 1$, we have

$$\begin{aligned} \sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s)) &\leq \sup_{-\tau_2 \leq s \leq 0} \mathbb{V}(x(s))E_\alpha(\tau_2 T^\alpha) + (E_\alpha(\tau_2 T^\alpha) - 1) \sup_{-\tau_2 \leq s \leq t-\tau_1} \mathbb{V}(x(s)) \\ &\quad + \gamma_0 E_\alpha(\tau_2 T^\alpha) \Gamma(\alpha) \sup_{s \in [0, T]} \mathcal{J}^\alpha \|\omega(s)\|^2. \end{aligned}$$

By induction and since the function $\sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s))$ is non-decreasing with respect to t , we obtain that

$$\sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s)) \leq \sup_{-\tau_2 \leq s \leq 0} \mathbb{V}(x(s)) \lambda_{\max}(P^{-1}E) + \beta_2^* \leq \beta_1 \|\varphi\|^2 + \beta_2^*, \quad t \in [0, T]. \quad (3.14)$$

where $\beta_2^* = \gamma_0 E_\alpha(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \Gamma(\alpha) \sup_{s \in [0, T]} \mathcal{J}^\alpha \|\omega(s)\|^2$ due to

$$\sup_{-\tau_2 \leq s \leq 0} \mathbb{V}(x(s)) \leq \lambda_{\max}(P^{-1}E) \|\varphi\|^2. \quad (3.15)$$

Consequently,

$$x(t)^\top P^{-1} E x(t) \leq \sup_{-\tau_2 \leq s \leq t} \mathbb{V}(x(s)) \leq \beta_1 \|\varphi\|^2 + \beta_2^*, \quad t \in [0, T].$$

Besides, since

$$x(t)^\top P^{-1} E x(t) \geq \lambda_{\min}(P_1) \|x_1(t)\|^2, \quad t \in [0, T],$$

which shows that if $\|\varphi\|^2 \leq c_1$ then

$$\|x_1(t)\|^2 \leq \frac{\beta_1 c_1 + \beta_2^*}{\lambda_{\min}(P_1)}, \quad t \in [0, T]. \quad (3.16)$$

To get the estimation of $\|x(t)\|^2$, we need to estimate the second state $\|x_2(t)\|$. For this, we describe the second equation of (3.2) as

$$x_2(t) = -(\mathbb{A}_{22})^{-1} \left[\mathbb{A}_{21} x_1(t) + D_{21} x_1(t - \tau(t)) + D_{22} x_2(t - \tau(t)) + \omega_2(t) \right].$$

Taking the estimation (3.16) into account, we get

$$\begin{aligned}\|x_1(t - \tau(t))\|^2 &\leq c_1 + \frac{\beta_1 c_1 + \beta_2^*}{\lambda_{\min}(P_1)} = \beta_3 c_1 + \beta_4^*, \\ \|(\mathbb{A}_{22})^{-1} \omega_2(t)\| &\leq \|(\mathbb{A}_{22})^{-1}\| \|W \omega(t)\| \leq \|(\mathbb{A}_{22})^{-1}\| \|W\| \sqrt{d} = \beta_7,\end{aligned}$$

where $\beta_4^* = \frac{\beta_2^*}{\lambda_{\min}(P_1)}$. Hence,

$$\|x_2(t)\| \leq \zeta^* + \beta \|x_2(t - \tau(t))\|, \quad t \in [0, T], \quad (3.17)$$

where $\zeta^* = 2\beta \sqrt{\beta_3 c_1 + \beta_4^*} + \beta_7$. On the other hand, using the inequality (3.17) for $t \in [0, \tau_1]$, we have $\|x_2(t)\| \leq \zeta^* + \beta \sqrt{c_1}$, because of $t - \tau_2 \leq t - \tau(t) \leq t - \tau_1 \leq 0$, and $\|x_2(t - \tau(t))\|^2 \leq c_1$. By induction, for $t \in [i\tau_1, (i+1)\tau_1] \cap [0, T]$, $i\tau_1 \leq T$, we get $\|x_2(t)\| \leq \zeta^* \sum_{k=0}^i \beta^k + \beta^{i+1} \sqrt{c_1}$, and hence

$$\|x_2(t)\| \leq \zeta^* \beta_5 + \beta_6 \sqrt{c_1}, \quad t \in [0, T]. \quad (3.18)$$

Finally, combining conditions (3.16), (3.18) with (3.5) and noting $\beta_2^* \leq \beta_2, \zeta^* \leq \zeta$, we obtain that

$$\begin{aligned}\|x(t)\|^2 &= \|x_1(t)\|^2 + \|x_2(t)\|^2 \leq \frac{\beta_2^* c_1 + \beta_3}{\lambda_{\min}(P_1)} + (\zeta^* \beta_5 + \beta_6 \sqrt{c_1})^2 \\ &\leq \frac{\beta_2 c_1 + \beta_3}{\lambda_{\min}(P_1)} + (\zeta \beta_5 + \beta_6 \sqrt{c_1})^2 \leq c_2, \quad t \in [0, T].\end{aligned}$$

Step 3. The γ -optimal level condition. From (3.12) and (3.14) it follows that

$$\begin{aligned}\|z(t)\|^2 &\leq 2\|Cx(t)\|^2 + 2\|Hx(t - \tau(t))\|^2 \\ &\leq \gamma_0 \|\omega(t)\|^2 + 2\tau_2(\beta_1 \|\varphi\|^2 + \beta_2^*) - \mathcal{D}_C^\alpha(\mathbb{V}(x(t))).\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathcal{J}^\alpha \|z(t)\|^2 &\leq \gamma_0 I^\alpha \|\omega(t)\|^2 + \mathcal{J}^\alpha (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) - \mathcal{J}^\alpha \mathcal{D}_C^\alpha(\mathbb{V}(x(t))) \\ &= \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{t^\alpha}{\Gamma(\alpha+1)} + [-\mathbb{V}(x(t)) + \mathbb{V}(x(0))] \\ &\leq \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{t^\alpha}{\Gamma(\alpha+1)} + \mathbb{V}(x(0)) \\ &\leq \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{t^\alpha}{\Gamma(\alpha+1)} + \mathbb{V}(x(0)) \\ &\leq \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + (2\tau_2 \beta_1 \|\varphi\|^2 + 2\tau_2 \beta_2^*) \frac{T^\alpha}{\Gamma(\alpha+1)} + \lambda_{\max}(P^{-1}E) \|\varphi\|^2 \\ &= \gamma_0 \mathcal{J}^\alpha \|\omega(t)\|^2 + \left(2\tau_2 \beta_1 \frac{T^\alpha}{\Gamma(\alpha+1)} + \lambda_{\max}(P^{-1}E)\right) \|\varphi\|^2 + 2\tau_2 \beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &= \gamma_0 (\mathcal{J}^\alpha \|\omega(t)\|^2 + c_0 \|\varphi\|^2) + 2\tau_2 \beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

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where $c_0 = \frac{\left(2\tau_2\beta_1 \frac{T^\alpha}{\Gamma(\alpha+1)} + \lambda_{\max}(P^{-1}E)\right)}{\gamma_0}$. Moreover, for all $t \in [0, T]$ we have

$$\sup_{t \in [0, T]} \mathcal{J}^\alpha \|z(t)\|^2 \leq \gamma_0 \left(\sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2 + c_0 \|\varphi\|^2 \right) + 2\tau_2\beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)},$$

which leads to

$$\begin{aligned} \frac{\sup_{t \in [0, T]} \mathcal{J}^\alpha \|z(t)\|^2}{c_0 \|\varphi\|^2 + \sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} &\leq \gamma_0 + \frac{2\tau_2\beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)}}{c_0 \|\varphi\|^2 + \sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} \\ &\leq \gamma_0 + \frac{2\tau_2\beta_2^* \frac{T^\alpha}{\Gamma(\alpha+1)}}{\sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} \\ &= \gamma_0 + 2\tau_2\gamma_0 E_{\alpha, \alpha}(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \Gamma(\alpha) \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &= \gamma_0 \left(1 + 2\tau_2 E_{\alpha, \alpha}(\tau_2 T^\alpha) \sum_{j=0}^{[T/\tau_1]} (E_\alpha(\tau_2 T^\alpha) - 1)^j \frac{T^\alpha}{\alpha} \right) = \gamma. \end{aligned}$$

Therefore, we finally obtain that

$$\frac{\|z(t)\|_\alpha^2}{c_0 \|\varphi\|^2 + \|\omega(t)\|_\alpha^2} = \frac{\sup_{t \in [0, T]} \mathcal{J}^\alpha \|z(t)\|^2}{c_0 \|\varphi\|^2 + \sup_{t \in [0, T]} \mathcal{J}^\alpha \|\omega(t)\|^2} \leq \gamma,$$

which completes the proof of the theorem. $\square$

REMARK 3.1 In Theorem 3.1, we assume that c_1, c_2, γ, d are given positive parameters. Therefore, to check the conditions of the theorem, we have to prescribe these parameters firstly. Since the matrix relation (3.3) is not a linear matrix inequality, but we can reduce it into a single strict LMI by some matrix transformation. Also, since the scalars c_1, c_2 are not involved in the LMI condition (3.4) we first find the unknowns of LMI (3.4) by using LMI Tollbox algorithm (Gahinet et al., 1985) and then verify the inequality (3.5).

REMARK 3.2 It is noted that in the fractional-order system context, the H_∞ finite-time controller are designed in (Zhang et al., 2014; Mathiyalagan and Balachandran, 2016; Liu et al., 2018; Osorio-Gorllio et al., 2018, Thuan et al., 2020) for the systems with or without delays. In this paper, the design method is proposed for the singular FDEs with nondifferentiable time-varying delay, which can not be applicable in the mentioned papers. Theorem 3.1 provides an analytical approach to the design of finite-time H_∞ controllers for singular FDEs time-delay systems and the obtained result can be considered as extensions of those in the mentioned papers to time-varying delay systems.

EXAMPLE 3.2 Consider the singular FDEs (2.1), where $\alpha = 0.1, d = 2$ and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, D = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$W = 10^{-15} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, C = 10^{-15} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, H = 10^{-15} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\tau(t) = 0.1 + 0.1|\sin t|,$$

We have $d = 2, \tau_1 = 0.1$. For $\gamma = 0.5, c_1 = 0.5, c_2 = 4, T = 20$, the solutions of LMI (3.4) can be determined as

$$P = \begin{pmatrix} 1979.3 & 0 \\ 0 & 112.4 \end{pmatrix}, \quad Y = \begin{pmatrix} 631.4239 & 11.2376 \\ 11.2376 & 11.2376 \end{pmatrix}.$$

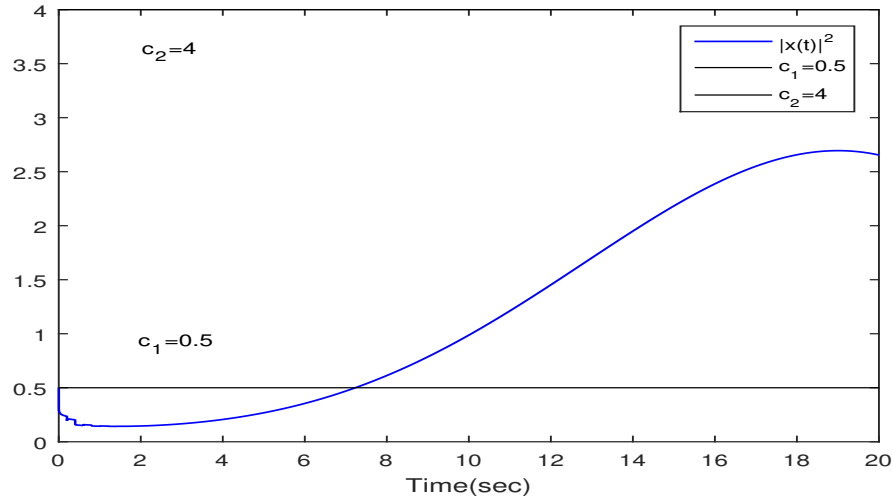


FIG. 1. The time history of $\|x(t)\|^2$

Hence,

$$L = Y(P^{-1})^\top = \begin{bmatrix} 0.3190 & 0.1000 \\ 0.0057 & 0.1000 \end{bmatrix}, \quad \mathbb{A} = A + BL = \begin{bmatrix} -0.4133 & -0.0000 \\ -0.0057 & -0.2000 \end{bmatrix}, \quad \mathbb{A}_{22} = -0.2000.$$

Moreover, we have

$$\gamma_0 = 0.1707, \beta = 0.5, \beta_1 = 0.0012, \beta_2 = 1.6462, \beta_3 = 1,$$

$$\beta_4 = 0.0008, \beta_5 = 2, \beta_6 = 0.5, \beta_7 = 1.4142e - 14, \xi = 0.7077,$$

for verifying the condition (3.5):

$$\left(\frac{\beta_2 c_1 + \beta_3}{\lambda_{\min}(P_1)} + [\zeta \beta_5 + \beta_6 \sqrt{c_1}]^2 \right) = 3.1301 \leq c_2 = 4.$$

Then, by Theorem 3.1 the finite-time H_∞ control of the system is solvable with the following state feedback controller

$$u(t) = \begin{pmatrix} 0.3190 & 0.1000 \\ 0.0057 & 0.1000 \end{pmatrix} x(t), \quad t \in [0, 20].$$

Figure 1 and Figure 2 show the time history of $\|x(t)\|^2$ and the state response of the system with the initial condition $\varphi(t) = [0.5, 0.5]^T$, respectively.

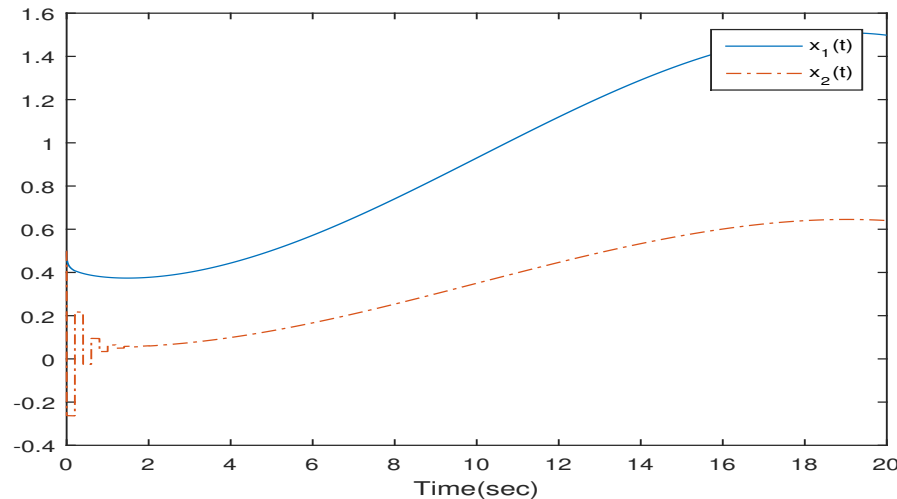


FIG. 2. The state response of the solutions

4. Conclusion

In summary, we have studied finite-time H_∞ control of linear singular FDEs with interval time-varying delay. Based on the fractional calculus and singular value decomposition approach, a set of new sufficient conditions are established for solving the problem that provides the designing the state feedback controllers and guaranteeing the resulting closed-loop system to be regular, impulse-free and finite-time stable with a given H_∞ performance index. The obtained results can be considered as extensions of those in various existing papers to time-varying delay systems.

Acknowledgments

The research of the first author is supported by the Chiang Mai University, Thailand. The research of the second and third authors is supported by National Foundation for Science and Technology Development, Vietnam (NAFOSTED 101.01-2021.01). The authors are grateful to anonymous reviewers for their valuable comments and suggestions, which allow us to improve the paper.

Conflict of interest

The authors declare that there is no conflict of interest.

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