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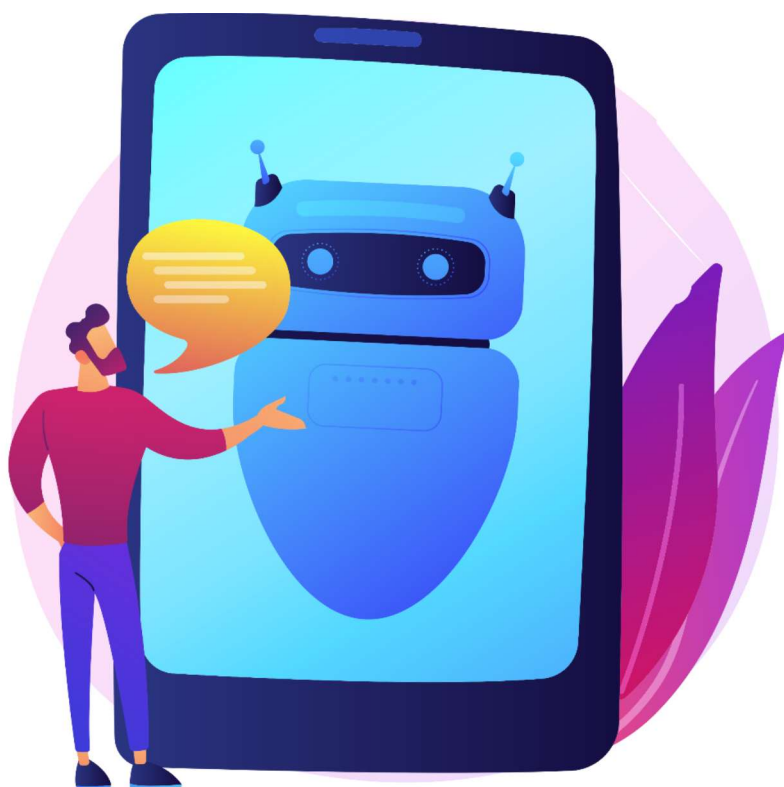
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ТЕХНИКА И ТЕХНОЛОГИЯ

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BENDING VIBRATIONS OF EULER–BERNOULLI BEAMS IN FLEXIBLE ROBOTIC MANIPULATORS

Abstract

Vibration is a common phenomenon in both natural and engineering systems. In modern applications, many real-world problems require highly accurate models, which necessitate the consideration of elastic vibrations in structural components. The analysis and control of such vibrations are therefore of critical importance. This paper investigates the bending vibrations of Euler–Bernoulli beams. The differential equations governing transverse vibrations are first derived. Based on these equations, the natural frequencies and corresponding mode shapes are determined. The forced vibration response is then analyzed to calculate the deflections at various points along the beam under different excitation frequencies. These results provide a foundation for studying the dynamic behavior of flexible robotic arms.

Keyword:

Euler–Bernoulli beam, bending vibrations, natural frequency, mode shape, flexible robot.

1. Introduction

Beams are among the most common structural models in engineering applications [1–3]. The study of beam behavior plays a crucial role in evaluating the load-bearing capacity and performance of structures. A wide variety of structures can be modeled as Euler–Bernoulli beams, including robotic manipulators [4]. In recent decades, the increasing demand for precision and high-speed operation of robotic end-effectors has led researchers to investigate the vibration characteristics of robot links. Consequently, the flexible links of robots are often modeled as Euler–Bernoulli beams [5,6]. This paper focuses on the investigation of the bending vibrations of Euler–Bernoulli beams, aiming to determine their natural frequencies and corresponding mode shapes.

2. Formulation of the Governing Equation for Beam Bending Vibrations

In the analysis of beam bending vibrations, it is assumed that the beam's length is significantly greater than its other dimensions, and that the cross-section of the beam is symmetric with respect to both principal axes. The beam under consideration is illustrated in Fig. 1.

Assume that cross-sections of the beam remain plain and perpendicular to the deflection axis during deformation. The geometric axis of the undeformed beam is assumed to be straight, and this axis is taken as the x-axis. The z-axis is defined as perpendicular to the x-axis. The beam is assumed to undergo bending vibration only in the z-direction.

Unlike in the static case, here the transverse displacement w , rotation angle φ , bending moment M , and shear force Q are functions of both spatial coordinate x and time t : $w(x, t)$; $\varphi(x, t)$; $M(x, t)$; $Q(x, t)$.

According to the theory of strength of materials, the relationship between the transverse displacement and the rotation angle is given by:

$$tg\varphi = \frac{\partial w(x, t)}{\partial x} \approx \varphi(x, t) \quad (1)$$

To formulate the differential equations governing bending vibrations, consider an infinitesimal beam element of length dx , as illustrated in Fig. 2.

The total rotation angle φ is assumed to consist of the bending-induced rotation θ (caused by the

bending moment M) and the shear-induced rotation (caused by the shear force Q):

$$\varphi = \frac{\partial w}{\partial x} = \psi + \theta \quad (2)$$

Applying d'Alembert's principle and considering the equilibrium of forces in the z -direction, we obtain:

$$-dm \frac{\partial^2 w}{\partial t^2} + Q + \frac{\partial Q}{\partial x} dx - Q + p(x,t)dx - \beta dm \frac{\partial w}{\partial t} = 0 \quad (3)$$

so that m represents the mass per unit length of the beam.

From the moment equilibrium of the element, we derive:

$$M + \frac{\partial M}{\partial x} dx - M - Q \frac{dx}{2} - \left(Q + \frac{\partial Q}{\partial x} dx \right) \frac{dx}{2} + dJ \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (4)$$

Here, dJ is the mass moment of inertia of the beam element about the y -axis (see Fig. 2), and I is the second moment of area of the cross section.

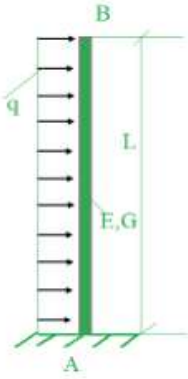


Figure 1 – Beam under bending deformation

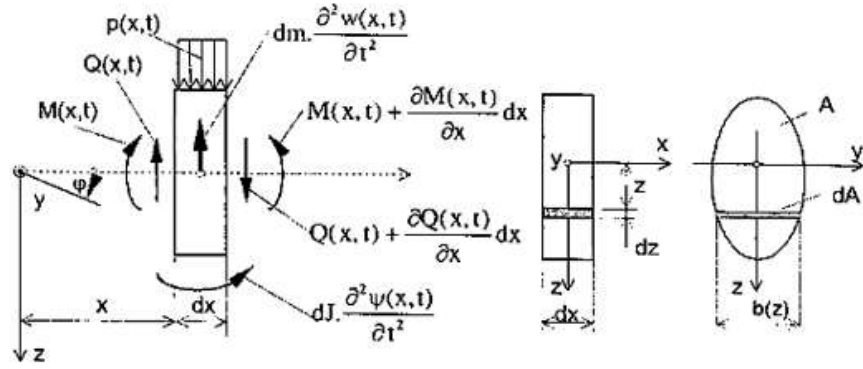


Figure 2 – Beam element

From Eqs. (3) and (4), we obtain:

$$\mu(x) \frac{\partial^2 w}{\partial t^2} = \frac{\partial Q}{\partial x} + p(x,t) - \beta \mu \frac{\partial w}{\partial t} \quad (5)$$

$$pI(x) \frac{\partial^2 \psi}{\partial t^2} = Q - \frac{\partial M}{\partial x} \quad (6)$$

According to classical strength of materials theory, the following constitutive relations hold:

$$M = -EI(x) \frac{\partial \Psi}{\partial x} \quad (7)$$

$$Q = k^* GA(x) \theta = k^* GA(x) \left(\frac{\partial w}{\partial x} - \Psi \right) \quad (8)$$

where E is Young's modulus, G is the shear modulus, and k^* is the shear correction factor.

Substituting expressions (7) and (8) into Eqs. (5) and (6) yields a coupled system of second-order partial differential equations in terms of the transverse displacement $w(x,t)$ and rotation angle $\psi(x,t)$

$$\mu(x) \frac{\partial^2 w}{\partial t^2} = k^* G \frac{\partial}{\partial x} \left[A(x) \left(\frac{\partial w}{\partial x} - \Psi \right) \right] + p(x,t) - \beta \mu \frac{\partial w}{\partial t} \quad (9)$$

$$pI(x) \frac{\partial^2 \psi}{\partial t^2} = k^* GA(x) \left(\frac{\partial w}{\partial x} - \Psi \right) + E \frac{\partial}{\partial x} \left[I(x) \frac{\partial \psi}{\partial x} \right] \quad (10)$$

To solve the above system, appropriate boundary and initial conditions are required.

Assuming the beam is homogeneous and has a constant cross-section, $A(x)$ and $I(x)$ are constants. Under this assumption, Eqs. (9) and (10) simplify to:

$$\mu(x) \frac{\partial^2 w}{\partial t^2} = k^* G^* A \frac{\partial}{\partial x} \left[\left(\frac{\partial w}{\partial x} - \Psi \right) \right] + p(x, t) - \beta \mu \frac{\partial w}{\partial t} \quad (11)$$

$$pI(x) \frac{\partial^2 \Psi}{\partial t^2} = k^* G A \left(\frac{\partial w}{\partial x} - \Psi \right) + EI \frac{\partial}{\partial x} \left[\frac{\partial \Psi}{\partial x} \right] \quad (12)$$

Differentiating Eq. (12) with respect to x , and then adding to Eq. (11), yields:

$$\mu \frac{\partial^2 w}{\partial t^2} + \beta \mu \frac{\partial w}{\partial t} - p(x, t) + EI \frac{\partial^3 \Psi}{\partial x^3} - \rho I \frac{\partial^3 \Psi}{\partial x \partial t^2} = 0 \quad (13)$$

From Eq. (11), we isolate:

$$\frac{\partial \Psi}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\mu}{Gk^* A} \frac{\partial^2 w}{\partial t^2} + \frac{1}{Gk^* A} p(x, t) - \frac{\beta \mu}{GkA} \frac{\partial w}{\partial t}$$

Taking the spatial derivative of this expression followed by two temporal derivatives, we obtain:

$$\begin{aligned} \frac{\partial^3 \Psi}{\partial x^3} &= \frac{\partial^4 w}{\partial x^4} - \frac{\mu}{Gk^* A} \frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{1}{Gk^* A} \frac{\partial^2 p(x, t)}{\partial x^2} - \frac{\beta \mu}{GkA} \frac{\partial^3 w}{\partial x^2 \partial t} \\ \frac{\partial^3 \Psi}{\partial x \partial t^2} &= \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\mu}{Gk^* A} \frac{\partial^4 w}{\partial t^4} + \frac{1}{Gk^* A} \frac{\partial^2 p(x, t)}{\partial t^2} - \frac{\beta \mu}{GkA} \frac{\partial^3 w}{\partial t^3} \end{aligned} \quad (14)$$

Substituting back into Eq. (13), we derive:

$$\begin{aligned} \mu \frac{\partial^2 w}{\partial t^2} - p(x, t) + \beta \mu \frac{\partial w}{\partial t} + EI \left[\frac{\partial^4 w}{\partial x^4} - \frac{\mu}{Gk^* A} \frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{1}{Gk^* A} \frac{\partial^2 p(x, t)}{\partial x^2} - \frac{\beta \mu}{GkA} \frac{\partial^3 w}{\partial x^2 \partial t} \right] \\ - pI \left[\frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\mu}{Gk^* A} \frac{\partial^4 w}{\partial t^4} + \frac{1}{Gk^* A} \frac{\partial^2 p(x, t)}{\partial t^2} - \frac{\beta \mu}{GkA} \frac{\partial^3 w}{\partial t^3} \right] = 0 \end{aligned} \quad (15)$$

Assuming a slender beam, the shear deformation due to transverse shear force is negligible, i.e., $k^* G \rightarrow \infty$. Noting that $m = \rho A$ is the mass per unit length, we simplify Eq. (15) to:

$$EI \frac{\partial^4 w}{\partial x^4} - pI \frac{\partial^4 w}{\partial t^2 \partial x^2} + \mu \frac{\partial^2 w}{\partial t^2} + \beta \mu \frac{\partial w}{\partial t} = p(x, t) \quad (16)$$

Equation (16) still includes the effect of rotary inertia. According to Rayleigh's studies, the contribution of rotary inertia becomes significant only at higher vibration modes. For low-frequency regions, the rotary inertia term can be neglected. Setting $pI=0$, we arrive at the classical Euler–Bernoulli beam vibration equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^2 w}{\partial t^2} + \beta \mu \frac{\partial w}{\partial t} = p(x, t) \quad (17)$$

Neglecting structural damping, the governing equation takes the form:

$$EI \frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^2 w}{\partial t^2} = p(x, t) \quad (18)$$

3. Free Bending Vibrations of a Homogeneous Euler–Bernoulli Beam with a Constant Cross - Section

We first consider the free bending vibration of a homogeneous beam with constant cross section based on the Euler–Bernoulli model. From the differential equation (18), the governing equation for free bending vibration is:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\mu}{EI} \frac{\partial^2 w}{\partial t^2} = 0 \quad (19)$$

Applying the method of separation of variables, we assume the solution of equation (19) to be:

$$w(x, t) = X(x)T(t) \quad (20)$$

Substituting equation (20) into (19), we obtain:

$$T(t)X^{(IV)}(x) + \frac{\mu}{EI} \ddot{T}(t)X(x) = 0 \quad (21)$$

Dividing both sides by $X(x)T(t)$, we get:

$$-\frac{\ddot{T}(t)}{T(t)} = \frac{EI}{\mu} \frac{X^{(IV)}(x)}{X(x)} \quad (22)$$

Since the left-hand side depends only on t , and the right-hand side depends only on x , both must be equal to a constant, denoted as ω^2 . This yields the system:

$$-\frac{\ddot{T}(t)}{T(t)} = \omega^2 \rightarrow \ddot{T}(t) + \omega^2 T(t) = 0 \quad (23)$$

$$\frac{EI}{\mu} \frac{X^{(IV)}(x)}{X(x)} = \omega^2 \rightarrow X^{(IV)}(x) - \frac{\mu\omega^2}{EI} X(x) = 0 \quad (24)$$

The solution to equation (23) is:

$$T(t) = A \cos \omega t + B \sin \omega t \quad (25)$$

To determine the natural frequencies, we need to solve equations (24). For convenience, we introduce the nondimensional quantity:

$$\lambda^4 = \frac{\mu\omega^2}{EI} l^4 \quad (26)$$

Equation (24) thus becomes:

$$X^{(IV)}(x) - \left(\frac{\lambda}{l}\right)^4 X(x) = 0 \quad (27)$$

The general solution to (27) is:

$$X(x) = C_1 \cos\left(\lambda \frac{x}{l}\right) + C_2 \sin\left(\lambda \frac{x}{l}\right) + C_3 \cosh\left(\lambda \frac{x}{l}\right) + C_4 \sinh\left(\lambda \frac{x}{l}\right) \quad (28)$$

The constants C_i are determined from the boundary conditions.

At the clamped end ($x=0$), both the deflection and rotation are zero:

$$X = 0, \quad \frac{dX}{dx} = 0 \quad (29)$$

At the free end ($x=l$), the bending moment and shear force are zero:

$$\frac{d^2 X}{dx^2} = 0, \quad \frac{d^3 X}{dx^3} = 0 \quad (30)$$

Thus, the boundary conditions are:

$$\begin{aligned} w(0, t) = 0, \quad M(l, t) = -EI w''(l, t) = 0 \\ w'(0, t) = 0, \quad Q(l, t) = -EI w'''(l, t) = 0 \end{aligned} \quad (31)$$

Substituting the general solution (28) into the boundary conditions, we obtain a system of four homogeneous linear equations:

$$\begin{aligned} X(0) = 0 & : C_1 + C_3 = 0 \\ X'(0) = 0 & : C_2 + C_4 = 0 \\ X''(l) = 0 & : -C_1 \cos \lambda - C_2 \sin \lambda + C_3 \cosh \lambda + C_4 \sinh \lambda = 0 \\ X'''(l) = 0 & : C_1 \sin \lambda - C_2 \cos \lambda + C_3 \sinh \lambda + C_4 \cosh \lambda = 0 \end{aligned} \quad (32)$$

From the first two equations, we find $C_1 = -C_3$, $C_2 = -C_4$ Substituting into the remaining equations yields:

$$\begin{aligned} C_1(\cos \lambda + \cosh \lambda) + C_2(\sin \lambda + \sinh \lambda) &= 0 \\ C_1(\sin \lambda - \sinh \lambda) - C_2(\cos \lambda + \cosh \lambda) &= 0 \end{aligned} \quad (33)$$

For a non-trivial solution to exist, the determinant of the coefficient matrix must be zero. This gives the characteristic equation:

$$\Delta = \begin{vmatrix} \cos \lambda + \cosh \lambda & \sin \lambda + \sinh \lambda \\ \sin \lambda - \sinh \lambda & -(\cos \lambda + \cosh \lambda) \end{vmatrix} = 0 \quad (34)$$

Solving equation (34) numerically or graphically yields an infinite set of eigenvalues $\lambda = \lambda_i \approx (2i-1)2\pi$ (see Fig. 3).

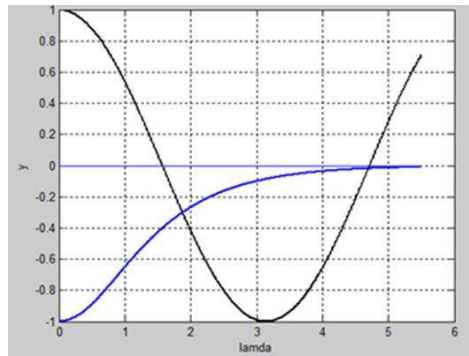


Figure 3 – Graphical Solution for the Eigenvalue λ

Once the eigenvalues λ_i , the natural frequencies can be computed from equation (26):

$$\omega_i = \frac{\lambda_i^2}{l^2} \sqrt{\frac{EI}{\mu}} \quad (35)$$

The structural properties of the beam are summarized in Table 1.

Table 1

Beam parameters			
Parameter	Symbol	Value	Unit
Flexural rigidity	EI	7.65692×10^9	$N \cdot m^2$
Young's modulus	E	3.15×10^{11}	N/m^2
Beam width	b	1.2	m
Beam height	h	0.8	m
Beam length	L	32.4	m
Second moment of area	$I = bh^3 / 12$	0.0512	m^4
Cross-sectional area	$A = bh$	0.96	m^2
Linear mass density	μ	2352	kg/m
Shear modulus	G	2×10^8	N/m^2

The first six natural frequencies are:

Table 2

Natural frequencies		
Mode	l	w
1	1.8751	6.0432
2	4.6941	37.8723
3	7.8548	106.0435
4	10.9955	207.8029
5	14.1372	343.5130
6	17.2788	513.1490

Each eigenvalue λ_i corresponds to a mode shape given by:

$$X_i(x) = C_1^{(i)} \cos \frac{\lambda_i x}{1} + C_2^{(i)} \sin \frac{\lambda_i x}{1} + C_3^{(i)} \cosh \frac{\lambda_i x}{1} + C_4^{(i)} \sinh \frac{\lambda_i x}{1} \quad (36)$$

From the boundary condition system (32), we find:

$$C_2^{(i)} = -\frac{\cos \lambda_i + \cosh \lambda_i}{\sin \lambda_i + \sinh \lambda_i} C_1^{(i)}$$

$$C_4^{(i)} = -C_2^{(i)} = \frac{\cos \lambda_i + \cosh \lambda_i}{\sin \lambda_i + \sinh \lambda_i} C_1^{(i)} \quad (37)$$

substituting $T_i(t)$ into the solution yields:

$$X_i(x) = C_1^{(i)} \left[\cos \frac{\lambda_i x}{1} - \cosh \frac{\lambda_i x}{1} + \frac{\cos \lambda_i + \cosh \lambda_i}{\sin \lambda_i + \sinh \lambda_i} \left(-\sin \frac{\lambda_i x}{1} + \sinh \frac{\lambda_i x}{1} \right) \right] \quad (38)$$

The first six mode shapes are shown as figure 4-9:

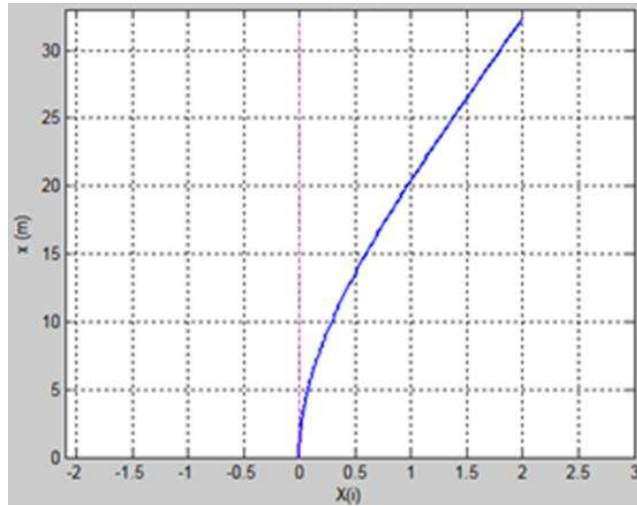


Figure 4 – Mode 1 Shape

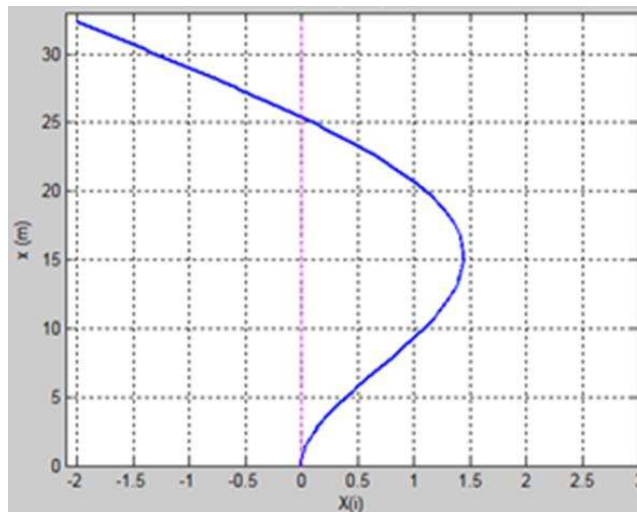


Figure 5 – Mode 2 Shape

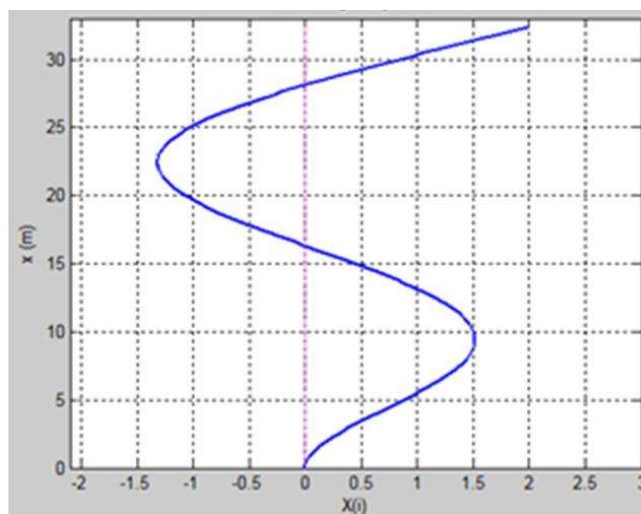


Figure 6 – Mode 3 Shape

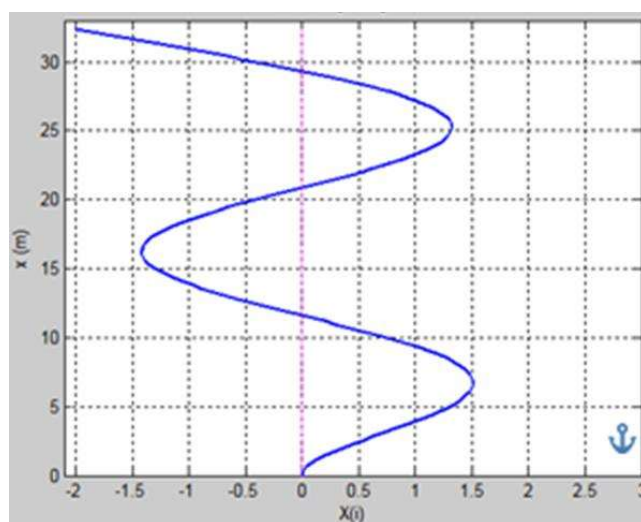


Figure 7 – Mode 4 Shape

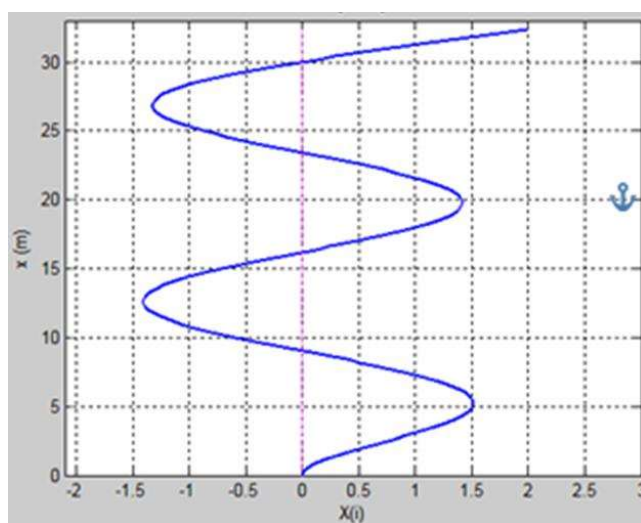


Figure 8 – Mode 5 Shape

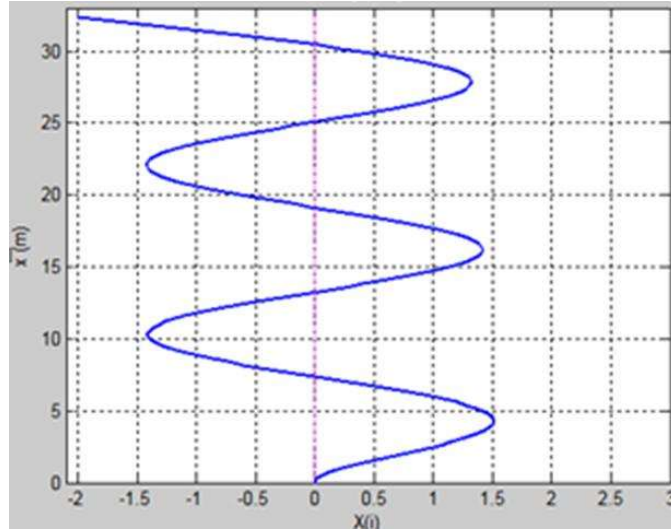


Figure 9 – Mode 6 Shape

The general solution of equation (20) for this case can be written as:

$$w(x, t) = \sum_{i=1}^{\infty} X_i(x) (A_i \cos \omega_i t + B_i \sin \omega_i t) \quad (39)$$

The constants A_i, B_i are determined by the initial conditions:

$$w(x, 0) = w_0(x), \quad \frac{\partial w(x, 0)}{\partial t} = v_0(x) \quad (40)$$

Therefore:

$$\sum_{i=1}^{\infty} X_i(x) A_i = w_0(x), \quad \sum_{i=1}^{\infty} \omega_i X_i(x) B_i = v_0(x) \quad (41)$$

Multiplying both sides by $X_k(x)$ and integrating from 0 to L, utilizing the orthogonality of the mode shapes:

$$A_k = \frac{\int_0^1 w_0(x) X_k(x) dx}{\int_0^1 X_k^2(x) dx}; \quad B_k = \frac{\int_0^1 v_0(x) X_k(x) dx}{\omega_k \int_0^1 X_k^2(x) dx} \quad (42)$$

4. Forced Bending Vibration of a Homogeneous Euler–Bernoulli Beam with Constant Cross-Section

The forced bending vibration of a homogeneous beam with constant cross-section is considered using the Euler–Bernoulli beam theory. The beam is subjected to an external force acting perpendicular to its longitudinal axis. The governing differential equation for the forced bending vibration of the Euler–Bernoulli beam is given in Equation (43), as derived from Equation (19)

$$EI \frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^2 w}{\partial t^2} = p(x, t) \quad (43)$$

The beam deflections corresponding to the first and sixth natural frequencies are obtained by solving the system of differential equations, as illustrated in Figures 10–12.

Deflection at Various Points at $x=L$, for the excitation frequency $\Omega=0.5$ and $\omega_1=3.02$

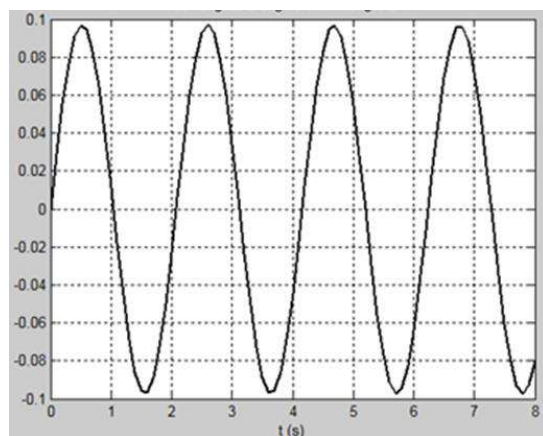


Figure 10 – Deflection at $x=L$ for the first mode shape

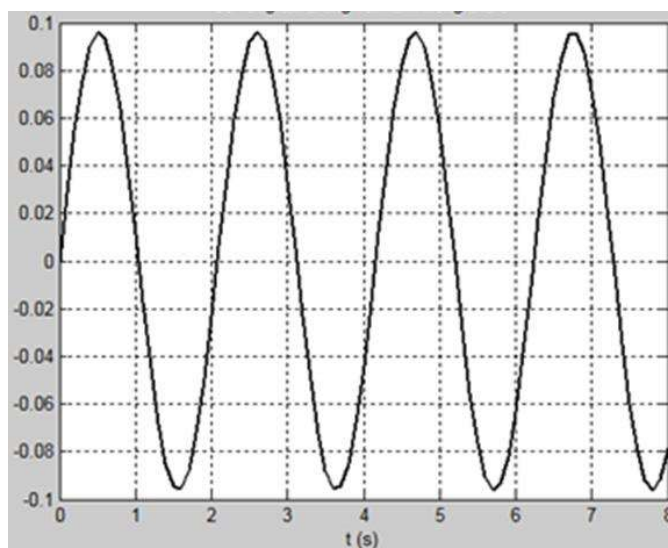


Figure 11 – Deflection at $x=L$ for the sixth mode shape

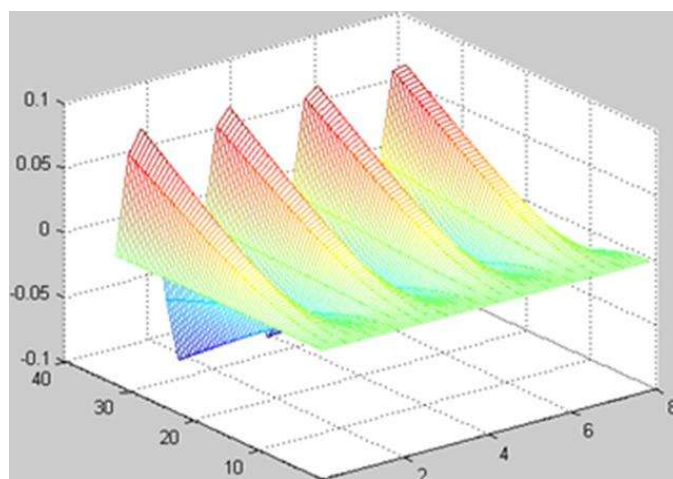


Figure 12 – Three-dimensional deflection at $x=L$ for the first mode shape

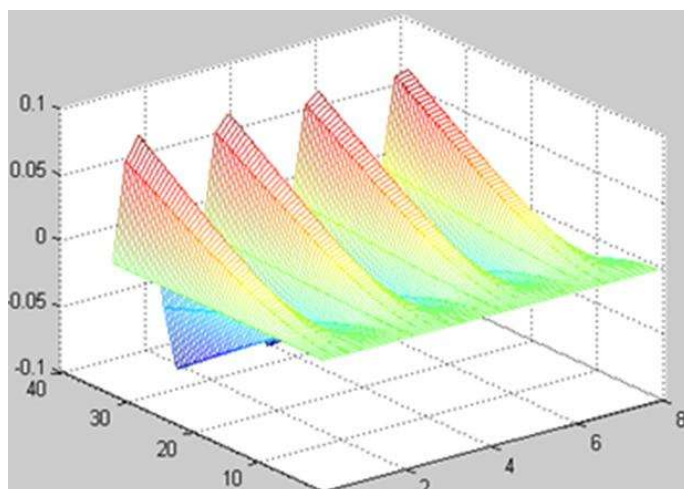


Figure 13 – Three-dimensional deflection at $x=L$ for the sixth mode shape
Deflection at $x=L$ for the Excitation Frequency $\Omega=\omega_1$, as illustrated in Figures 14 –16.

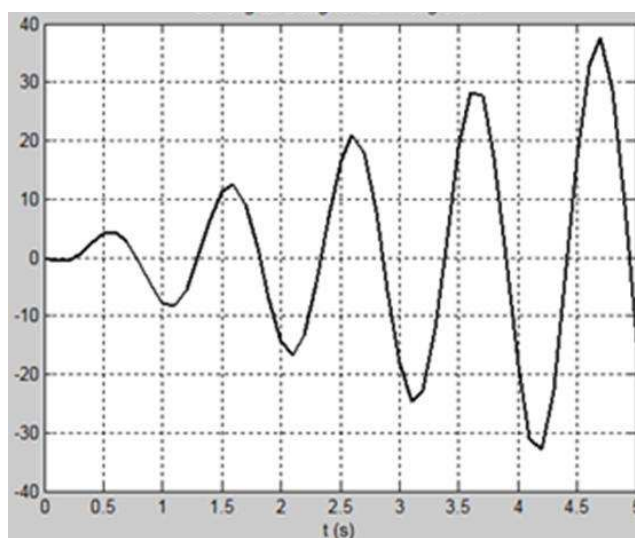


Figure 14 – Deflection at $x=L$ for the first mode shape

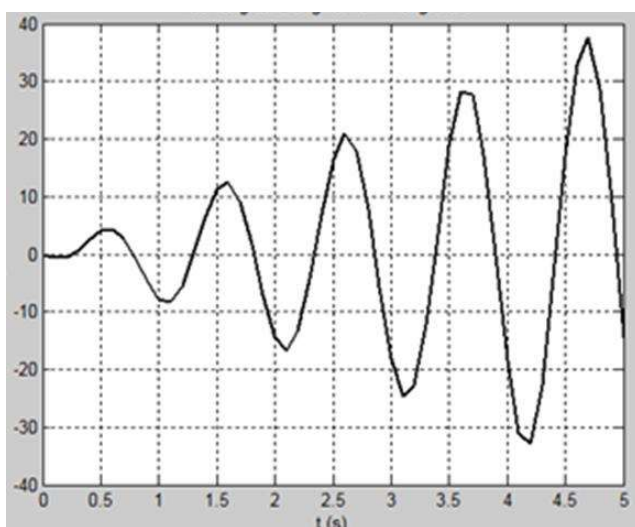


Figure 15 – Deflection at $x=L$ for the sixth mode shape

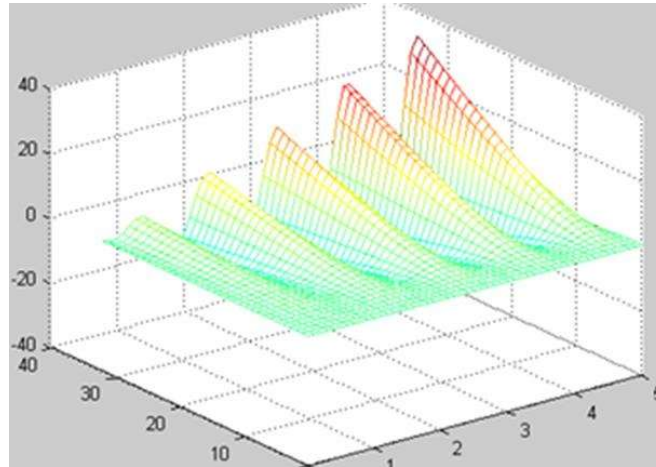


Figure 16 – Three-dimensional deflection at $x=L$ for the first mode shape

5. Conclusions

Based on the proposed modeling approach, the differential equations governing the bending vibrations of a cantilevered Euler–Bernoulli beam have been successfully derived. Numerical methods were employed to determine the natural frequencies and corresponding mode shapes of the system. These findings provide a theoretical foundation for analyzing and evaluating the dynamic response of flexible robotic arms modeled using Euler–Bernoulli beam theory.

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