



Effective Poroelastic Properties of N-Layered Composite Sphere Assemblage with Imperfect Interface: Application to Oolitic Rocks

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Abstract

This paper aims at deriving effective poroelastic properties of an N-layered composite sphere assemblage with an imperfect interface between two consecutive layers. The Linear Spring Model (LSM) is used to describe the interface behavior, in which the stress is continuous while the displacement is discontinuous across the interface. Each layer is a porous medium, and the porosity is assumed to be connected throughout all the layers. The derivation of effective poroelastic properties is based on thermoelastic solution derived by Hashin (Hashin 1991) (for 2 layers with imperfect interface) and generalized self-consistent homogenization scheme. The closed-form solutions of the Biot coefficient and the solid Biot modulus are derived, which are the main contribution of this study.

The solution obtained is applied to estimate the poroelastic properties of various oolitic rocks, which are constituted by an assemblage of grains (oolites) coated by a porous cement matrix. A two-step homogenization is proposed. The first step consists in the transition from the microscopic to the mesoscopic scale, where the effective properties of each phase (oolite and matrix) are estimated by using an iterative self-consistent scheme. In the second step (also called meso-macro transition/upscaling), the two phases oolite and embedding matrix are homogenized in the framework of the three-phase CSA model with interfaces. Comparisons with measurement data from various limestones are made. A sensitive study is performed to assess the role of imperfect interface parameters and to propose a reference set of parameters for all the considered materials.

Highlights

- Closed-form poroelastic solutions for n-layered spheres with imperfect interfaces.
- The Linear Spring Model is used to describe the interfacial displacement jump.
- A two-step upscaling scheme estimates effective properties of oolitic rocks.
- A sensitivity study results in a reference set of parameters for the ooid, the sparite matrix, and the interface for various oolitic limestones.

Keywords N-layered composite sphere · CSA · Imperfect interface · Linear spring model · Oolitic limestone · Poroelastic properties

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1 Introduction

Many materials exhibit the microstructure of an assemblage of porous grains coated by one or several porous layers (referred to as the matrix), such as oolitic rock (Fabre and Gustkiewicz 1997; Lion 2004; Lion et al. 2004; Trieu et al. 2020; Nguyen 2010; Hart and Wang 1995), iron ore (Grgic 2001), cement-based materials (Christensen and Lo 1979; Hashin and Monteiro 2002; Bary and Béjaoui 2006), etc. In such materials, the interface between grain and matrix, so-called the interfacial transition zone (ITZ), often present a high porosity. There are two approaches for representing ITZ: either by a thin layer or by an imperfect interface.

Hashin (Hashin 1962) is the pioneer work to introduce the composite sphere assemblage (CSA) model to estimate the elastic properties of these materials, where each grain is assumed to be spherical and surrounded by a spherical annulus matrix. Christensen and Lo (Christensen and Lo 1979) proposed a three-phase CSA model to obtain the explicit solution of the shear modulus. This 3-phase CSA model, also known as the generalized self-consistent model, was built based on the assumption that there exists an equivalent homogenized phase surrounding all composite sphere elements. Then, Hervé and Zaoui (Hervé and Zaoui 1993) extended the three-phase CSA model to an ' $n + 1$ '-phase model by deriving the analytical elastic stress and strain solutions of an n -layered spherical inclusion within an infinite medium subjected to far field stress or strain loading. Bary and Béjaoui (Bary and Béjaoui 2006) derived the effective diffusivity and the Biot coefficient for an n -layered composite sphere assemblage. However, in their solution, the effective properties of the whole medium are upscaled by $n-1$ times of the three-phase model. Trieu et al. (Trieu et al. 2022) derived the effective poroelastic properties (bulk and shear drained moduli, Biot coefficient and solid Biot modulus) of the n -layered composite assemblage in the framework of the ' $n + 1$ ' phase CSA model. In their models, the interface between the grain and the surrounding matrix is often represented by a thin weak layer, often called by interfacial transition zone (ITZ) (Trieu et al. 2022; Bui et al. 2021; Nguyen et al. 2011, 2019; Giraud et al. 2012; Pham et al. 2021a, 2020, 2021b; Lutz et al. 1997) and the interface between two layers is perfect bonding.

The debonding and imperfect interface is also an alternative approach for representing the highly porous and/or degraded interfacial transition zone (ITZ) identified around the grains. The effect of imperfect interface on composite mechanical and thermal behavior has already been investigated by numerous studies (Hashin 1991, 1990, 2002; Benveniste 1985; Achenbach and Zhu 1989, 1990; Duan et al. 2006, 2007a, 2007b; Duan and Karihaloo 2007). Two models have been often used to describe the imperfect interface

behavior of composites and polycrystalline materials. The first one is the Linear Spring Model (LSM), which is based on the assumption that tensions are continuous but displacements are discontinuous across the interface (Hashin 1991, 1990, 2002; Benveniste 1985; Achenbach and Zhu 1989, 1990). The second one is the interface stress model (ISM), in which it is assumed that the radial and the tangential stresses are discontinuous but displacements are continuous across the interface (Duan et al. 2006, 2007a, 2007b; Duan and Karihaloo 2007). The LSM model has been widely used. Hashin (Hashin 1991, 1990, 2002) used the LSM to estimate the effective elastic properties and thermal expansion coefficients for the cases of a spherical inclusion and a fiber composite with normal and shear imperfect interface. Moreover, Hashin (Hashin 1991, 1990) showed the equivalent between the spring and a thin flexible interphase layer, and the imperfect interface parameters can be determined from the interphase properties and thickness. Hervé-Luanco (Herve-Luanco 2014) proposed a procedure to deal with n -layered inclusion based composites with imperfect interface and derived its effective bulk and shear moduli using the generalized self consistent homogenization approach. To the best of the authors' knowledge, a closed-form poroelastic solutions of the Biot coefficient and the solid Biot modulus for the case of an n -layered concentric spherical assemblage with imperfect interfaces does not yet exist.

This study aims to derive the closed-form solutions of effective poro-elastic properties, including the Biot coefficient and the solid Biot modulus, of n -layered composite sphere assemblage with imperfect interfaces between two consecutive layers in the framework of ' $n + 1$ ' phase CSA model. The interface response is described via the Linear Spring Model (LSM). The derivation of the bulk and shear drained moduli are also provided. To do so, a n -layered concentric spherical assemblage is assumed to be surrounded by an $(n + 1)^{\text{th}}$ layer with unknown effective properties of n layers and subjected to a macroscopic strain at infinity and a pore pressure field within the whole domain. This problem is split into two subproblems. The first one corresponds to the drained condition (i.e. variation in pore pressure is nil) under the far field condition of the macroscopic strain, which allows determining the bulk and shear drained moduli. The mathematical procedure of this first subproblem is similar to Hervé-Luanco (Herve-Luanco 2014). The second one consists in the porous medium subjected to a pore pressure under the volume constant condition and results in the Biot coefficient and the solid Biot modulus.

The present solution is then applied to study the poroelastic properties of various oolitic limestones. The microstructure of considered rocks consists of quasi-spherical ooids coated by sparite cement matrix. Macropore are almost entirely located on the envelope of the oolitic grain, which is considered as an imperfect interface in this work.

Moreover, ooids and cement matrix are also constituted by solid phase and micropores. Therefore, a two-step microscopic-mesoscopic and mesoscopic-macroscopic upscaling model is proposed. The first step consists in homogenizing the properties of ooids and surrounding matrix by using an iterative self-consistent scheme. The second step uses 3-phase CSA model with imperfect interface to estimate the effective poroelastic properties of oolitic rocks. Comparisons with test data are made to demonstrate the ability of the proposed model to accurately predict the poroelastic properties of the studied rocks and to capture the evolution of these properties as a function of porosity. Furthermore, a reference set of constitutive parameters is proposed for various oolitic limestones, facilitating the application of the present model to these materials.

1.1 Notations

Subsequently, the following notations are considered

- $\underline{a}, \underline{b}$ are first order tensors
- $\underline{\underline{a}}, \underline{\underline{b}}$ are second order tensors
- $\underline{\underline{\underline{A}}}, \underline{\underline{\underline{B}}}$ are fourth order tensors
- $\underline{\underline{I}}$ is the second-order identity tensor
- $\underline{\underline{\underline{I}}}, \underline{\underline{\underline{J}}}$ and $\underline{\underline{\underline{K}}}$ are respectively the fourth-order symmetric identity tensor, spherical and deviatoric tensors
- δ_{ij} denotes Kronecker delta symbol, $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise
- $\otimes$ and $:$ are respectively the tensor product and contracted products on two indices

$$\underline{a} \otimes \underline{b} = a_i b_j e_i \otimes e_j$$

$$\underline{a} : \underline{b} = a_{ij} b_{ji}$$

$$\underline{\underline{\underline{A}}} : \underline{\underline{\underline{B}}} = A_{ijkl} B_{lkmn} e_i \otimes e_j \otimes e_m \otimes e_n$$

$$\underline{\underline{I}} = \delta_{ij} e_i \otimes e_j \quad (1)$$

$$\underline{\underline{\underline{I}}}_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$\underline{\underline{\underline{J}}}_{ijkl} = \frac{1}{3} \delta_{ij} \delta_{kl}$$

$$\underline{\underline{\underline{K}}}_{ijkl} = \underline{\underline{\underline{I}}}_{ijkl} - \underline{\underline{\underline{J}}}_{ijkl}$$

The brackets refer to an intrinsic average, $\langle \underline{a} \rangle$ represents the intrinsic average of $\underline{a}$ over the RVE denoted $\bar{\Omega}$, $\langle \underline{a} \rangle_\alpha$ represents the intrinsic average of $\underline{a}$ over the α phase

$$\langle \underline{a} \rangle = \frac{1}{\bar{\Omega}} \int_{\bar{\Omega}} \underline{a} d\Omega, \quad \langle \underline{a} \rangle_\alpha = \frac{1}{\Omega_\alpha} \int_{\Omega_\alpha} \underline{a} d\Omega \quad (2)$$

1.2 N-Phase CSA Model with Imperfect Interfaces for Poroelastic Properties

We consider a composite constituted by an n -layered spherical inclusion with imperfect bonding at the interfaces between two consecutive layers. As a reminder, the ' $n+1$ '-phase model deals with n -layered spherical inclusion embedded in an equivalent medium (i.e. $(n+1)^{\text{th}}$ -layer). Each layer (number i) or the phase i is assumed to be porous and isotropic, which is characterized physically by the drained bulk and shear moduli k_i, μ_i , the Biot coefficient b_i and the solid Biot modulus N_i . The phase i is limited by two spherical shells with the radius r_{i-1} and r_i ($1 \leq i \leq n$). The porosity is assumed to be connected through all the layers.

The interface behavior is governed by the Linear Spring Model (LSM). At the interface Γ_i between two layers i and $i+1$, the stress components are continuous, while the displacement components are discontinuous and proportional to the corresponding stress components (Hashin 1991).

$$\begin{aligned} \sigma_{rr}^{i+1} &= \sigma_{rr}^i = D_{rr}^i [\xi_r^i], \quad [\xi_r^i] = \xi_r^{i+1} - \xi_r^i \\ \sigma_{r\theta}^{i+1} &= \sigma_{r\theta}^i = D_{r\theta}^i [\xi_\theta^i], \quad [\xi_\theta^i] = \xi_\theta^{i+1} - \xi_\theta^i \\ \sigma_{r\varphi}^{i+1} &= \sigma_{r\varphi}^i = D_{r\varphi}^i [\xi_\varphi^i], \quad [\xi_\varphi^i] = \xi_\varphi^{i+1} - \xi_\varphi^i \end{aligned} \quad \text{with } i = (1, n-1) \quad (3)$$

where $D_{rr}^i, D_{r\theta}^i, D_{r\varphi}^i$ are spring constant parameters; σ_{mn}^i and $[\xi_l^i]$ are respectively the stress and the displacement jump across the interface Γ_i (with $mn = rr, r\theta, r\varphi$ and $l = r, \theta, \varphi$).

As phase $n+1$ is the effective homogeneous medium of n -layered composite sphere assemblage, the interface conditions at Γ_n is given by

$$\begin{aligned} \sigma_{rr}^{n+1} &= \sigma_{rr}^n, \quad \xi_r^{n+1} = \xi_r^n \\ \sigma_{r\theta}^{n+1} &= \sigma_{r\theta}^n, \quad \xi_\theta^{n+1} = \xi_\theta^n \\ \sigma_{r\varphi}^{n+1} &= \sigma_{r\varphi}^n, \quad \xi_\varphi^{n+1} = \xi_\varphi^n \end{aligned} \quad (4)$$

We consider a heterogeneous Representative Elementary Volume (REV) Ω bounded by its boundary $\partial\Omega$ and subjected to a uniform strain $\underline{\underline{E}}$ at the infinity and pore pressure field

p_l within the pores. As a reminder, the characteristic length scale of heterogeneities is small compared to the REV size. The poro-elastic field (stress, strain, displacement, pore pressure) is the solution of the following problem (Ulm et al. 2005):

$$\begin{aligned} \underline{\underline{\sigma}} &= \underline{\underline{\mathbb{C}}} : \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}}^p \text{ in } \Omega \\ \underline{\underline{\mathbb{C}}} &= \underline{\underline{\mathbb{C}}}_i \text{ in } \Omega_i \\ \underline{\underline{\sigma}}^p &= -b_i p_l \underline{\underline{I}} \text{ in } \Omega_i \\ \underline{\underline{\xi}} &= \underline{\underline{E}} \cdot \underline{\underline{x}} \text{ on } \partial\Omega \end{aligned} \quad (5)$$

where $\underline{\underline{\mathbb{C}}}_i, b_i$ are the stiffness tensor and the Biot coefficient of the phase i .

Problem (5) under loading $\begin{pmatrix} E, p_l \\ - \end{pmatrix}$ can be split into two sub

problems (P') and (P'').

The first subproblem (P') corresponds to drained conditions (i.e., $\sigma^p = 0, p_l = 0$) combining with the macroscopic strain tensor E :

$$\begin{aligned} \underline{\underline{\sigma}}' &= \underline{\underline{C}} : \underline{\underline{\varepsilon}}' \text{ in } \Omega \\ \underline{\underline{C}} &= \underline{\underline{C}}_i \text{ in } \Omega_i \\ \underline{\underline{\xi}}' &= \underline{\underline{E}}_{,x} \text{ on } \partial\Omega \end{aligned} \quad (6)$$

The second subproblem (P'') is imposed by the pore pressure p_l within the pore network and zero-displacement on its boundary problem $\begin{pmatrix} E = 0 \\ - \end{pmatrix}$:

$$\begin{aligned} \underline{\underline{\sigma}}'' &= \underline{\underline{C}} : \underline{\underline{\varepsilon}}'' + \underline{\underline{\sigma}}^p \text{ in } \Omega \\ \underline{\underline{C}} &= \underline{\underline{C}}_i \text{ in } \Omega_i \\ \underline{\underline{\sigma}}^p &= -b_i p_l \underline{\underline{i}} \text{ in } \Omega_i \\ \underline{\underline{\xi}}'' &= 0 \text{ on } \partial\Omega \end{aligned} \quad (7)$$

The problem (P') allows determining the drained bulk and shear moduli thank to the full drained condition ($p_l = 0$), while the Biot coefficient and Biot modulus are derived from the second problem (P''). The procedure to derive of the drained bulk and shear moduli was originally developed by Hervé-Luanco (Herve-Luanco 2014) and is reproduced in the Appendix to ensure the paper remains self-contained and self-consistent. This section only focuses on the deriv the closed-form solution of the Biot coefficient and the Biot modulus, which are the main contribution of this paper. The problem will be formulated in the spherical coordinate system. The macroscopic strain $\underline{\underline{E}}$

$$\xi_i(x)_{x \rightarrow \infty} = E_{ij} x_j \quad (8)$$

where E_{ij} are constant and symmetric and can be split into spherical and deviatoric

$$E_{ij} = E_{ij}^{iso} \delta_{ij} + E_{ij}^{dev} \quad (9)$$

1.3 Effective Biot Coefficient

Considering the second subproblem (P'') (Eq. (7)), where the pore pressure field p_l is imposed within the medium and a nil displacement is prescribed at the infinity ($\underline{\underline{E}} = 0$). The solutions for the non-zero displacement and the corresponding stress are derived in the same way of the derivation for the bulk modulus as described in the sub-section A.1 (see Eq. (54))

$$\begin{aligned} \xi_r^i &= H_i r + \frac{L_i}{r^2} \\ \sigma_{rr}^i &= 3H_i k_i - \frac{4L_i \mu_i}{r^3} - b_i p_l \end{aligned} \quad (i = 1, n+1) \quad (10)$$

The coefficient L_1 must vanish to avoid the singularity at the origin, and $H_{n+1} = 0, L_{n+1} = 0$ to satisfy the zero-radial displacement at infinity.

By introducing a poro-elastic parameter ω_r^i which denotes the discontinuity of radial displacement associated with a loading defined by the stress only caused by the fluid pressure $\sigma^p = -b_i p_l \underline{\underline{i}}$, Eqs. (3) and (4) defining the interface conditions can be written as

$$\begin{cases} \xi_r^{n+1}(r_n) = \xi_r^n(r_n) \\ \sigma_{rr}^{n+1}(r_n) = \sigma_{rr}^n(r_n) \end{cases} \text{ at interface } \Gamma_n \quad (11)$$

$$\begin{cases} \xi_r^{i+1}(r_i) = \xi_r^i(r_i) + \frac{\sigma_r^i(r_i)}{D_r^i} + \omega_r^i p_l \\ \sigma_{rr}^{i+1}(r_i) = \sigma_{rr}^i(r_i) \end{cases} \text{ at interface } \Gamma_i \text{ with } i = (1, n-1) \quad (12)$$

or in matrix form

$$\begin{aligned} I_{n+1}(r_n) \cdot V_{n+1} &= I_n(r_n) \cdot V_n + Y_n p_l \\ I_{i+1}(r_i) \cdot V_{i+1} &= [I_i(r_i) + J_i(r_i)] \cdot V_i + Y_i p_l \text{ with } i = (1, n-1) \end{aligned} \quad (13)$$

where

$$\begin{aligned} I_i(r) &= \begin{bmatrix} r & \frac{1}{r^2} \\ 3k_i & -4\frac{\mu_i}{r^3} \end{bmatrix}, J_i(r) = \begin{bmatrix} \frac{3k_i}{D_r^i} & -4\frac{\mu_i}{D_r^i r^3} \\ 0 & 0 \end{bmatrix}, \\ V_i &= \begin{bmatrix} H_i \\ L_i \end{bmatrix}, V_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix}, V_{n+1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ Y_i &= \begin{bmatrix} \left(\omega_r^i - \frac{b_i}{D_r^i}\right) \\ (b_{i+1} - b_i) \end{bmatrix}, Y_n = \begin{bmatrix} 0 \\ (b_{n+1} - b_n) \end{bmatrix} \end{aligned} \quad (14)$$

and then

$$\begin{aligned} 0 &= I_n(r_n) \cdot V_n + Y_n p_l \\ V_n &= ([N_{n-1} \dots N_2, N_1] \cdot V_1 + ([N_{n-1} \dots N_2] \cdot T_1 + \dots + N_{n-1} \cdot T_{n-2})) p_l + T_{n-1} p_l \\ &= \left(\prod_{i=1}^{n-1} N_i\right) \cdot V_1 + \sum_{i=1}^{n-2} \left(\prod_{j=i+1}^{n-1} N_j\right) \cdot T_i p_l + T_{n-1} p_l \end{aligned} \quad (15)$$

with

$$\begin{aligned} N_i &= [I_{i+1}(r_i)]^{-1} \cdot [I_i(r_i) + J_i(r_i)] \text{ with } i = (1, n-1) \\ T_i &= [I_{i+1}(r_i)]^{-1} \cdot Y_i \end{aligned} \quad (16)$$

The equilibrium condition at $r = r_n$ is given by relation

$$\begin{aligned} -Y_n p_l &= I_n(r_n) \cdot \left(\left(\prod_{i=1}^{n-1} N_i\right) \cdot V_1 + \sum_{i=1}^{n-2} \left(\prod_{j=i+1}^{n-1} N_j\right) \cdot T_i p_l + T_{n-1} p_l \right) \\ &= I_n(r_n) \cdot (Q_{n-1} \cdot V_1 + G_{n-1} p_l + T_{n-1} p_l) \end{aligned} \quad (17)$$

With

$$\begin{aligned} \mathbf{I}_n(r_n) &= \begin{bmatrix} r_n & \frac{1}{r_n^2} \\ 3k_n & -4\frac{\mu_n}{r_n^3} \end{bmatrix}, \mathbf{Q}_{n-1} = \prod_{i=1}^{n-1} \mathbf{N}_i = \begin{bmatrix} \mathbf{Q}_{11}^{n-1} & \mathbf{Q}_{12}^{n-1} \\ \mathbf{Q}_{21}^{n-1} & \mathbf{Q}_{22}^{n-1} \end{bmatrix} \\ \mathbf{G}_{n-1} &= \sum_{i=1}^{n-2} \left(\prod_{j=i+1}^{n-1} \mathbf{N}_j \right) \cdot \mathbf{T}_i = \begin{bmatrix} \mathbf{G}_{11}^{n-1} \\ \mathbf{G}_{21}^{n-1} \end{bmatrix}, \\ \mathbf{T}_{n-1} &= [\mathbf{I}_n(r_{n-1})]^{-1} \cdot \mathbf{Y}_{n-1} = \begin{bmatrix} \mathbf{T}_{11}^{n-1} \\ \mathbf{T}_{21}^{n-1} \end{bmatrix} \end{aligned} \quad (18)$$

It can be noticed that the unknown b_{n+1} only contributes to the matrix $\mathbf{Y}_n$. Moreover, Three matrix $\mathbf{Q}_{n-1}$, $\mathbf{G}_{n-1}$ and $\mathbf{T}_{n-1}$ only associate with the interface conditions at Γ_i with $i = (1, n-1)$ in which the properties of each phase are well known. The relation (17) may be written in the form:

$$\begin{bmatrix} 0 \\ -(b_{n+1} - b_n)p_l \end{bmatrix} = \begin{bmatrix} r_n & \frac{1}{r_n^2} \\ 3k_n & -4\frac{\mu_n}{r_n^3} \end{bmatrix} \cdot \left(\begin{bmatrix} \mathbf{Q}_{11}^{n-1} & \mathbf{Q}_{12}^{n-1} \\ \mathbf{Q}_{21}^{n-1} & \mathbf{Q}_{22}^{n-1} \end{bmatrix} \cdot \begin{bmatrix} H_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{11}^{n-1} p_l \\ \mathbf{G}_{21}^{n-1} p_l \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{11}^{n-1} p_l \\ \mathbf{T}_{21}^{n-1} p_l \end{bmatrix} \right) \quad (19)$$

Finally, solving linear equation system (19) with two unknowns (H_1, b_{n+1}) results in the effective Biot coefficient modulus b_{n+1} of *n-layered CSA*.

$$b_{n+1} = b_n + \frac{(3k_n + 4\mu_n)(\mathbf{Q}_{11}^{n-1}(\mathbf{G}_{21}^{n-1} + \mathbf{T}_{21}^{n-1}) - \mathbf{Q}_{21}^{n-1}(\mathbf{G}_{11}^{n-1} + \mathbf{T}_{11}^{n-1}))}{(r_n^3 \mathbf{Q}_{11}^{n-1} + \mathbf{Q}_{21}^{n-1})} \quad (20)$$

Although alternative approaches based on Levin's theorem could be considered for the evaluation of the homogenized Biot coefficient, the present formulation adopts a direct resolution of problem (P'') in order to ensure a consistent derivation of both b_{hom} and the solid Biot modulus N_{hom} , especially in the presence of imperfect interfaces.

1.4 Effective Solid Biot Modulus

In principle, the solid Biot modulus N_{hom} could be evaluated from the variation of porosity (or fluid content) obtained from the solution of problem P'' for a heterogeneous porous material. However, in the present multilayered configuration with imperfect interfaces, the presence of displacement discontinuities makes the direct evaluation of macroscopic porosity changes from local fields non-trivial. For this reason, the energy-based formulation above is adopted, which provides a consistent framework for deriving N_{hom} in heterogeneous poroelastic media with interface effects. An alternative approach based on average porosity changes could be envisaged by representing the interface as an equivalent interphase with effective porosity, although this is beyond the scope of the present work.

From the solution of subproblem (P'') $\left(\underline{\underline{E}} = 0, p_l \neq 0 \right)$,

$$\underline{\underline{\Sigma}} = \mathbb{C}_{hom} : \underline{\underline{E}} - b_{hom} p_l \underline{\underline{i}} \text{ in } \Omega \quad (21)$$

the effective energy $\mathcal{W}_{hom}$ of a REV (Ω) can be written by (see Willis (Willis and Yih 1981), Vincent et al. (Vincent et al. 2009) and Ulm et al. (Ulm et al. 2005))

$$\begin{aligned} \mathcal{W}_{hom} \left(\begin{bmatrix} \underline{\underline{E}} \\ p_l \end{bmatrix} \right) &= \frac{1}{2} \underline{\underline{E}} : \mathbb{C}_{hom} : \underline{\underline{E}} - p_l \underline{\underline{b}}_{-hom} : \underline{\underline{E}} - \frac{1}{2} \frac{p_l^2}{N_{hom}} \\ \mathcal{W}_{hom} \left(\begin{bmatrix} 0 \\ p_l \end{bmatrix} \right) &= -\frac{1}{2} \frac{p_l^2}{N_{hom}} \end{aligned} \quad (22)$$

where N_{hom} denote the effective solid Biot modulus. According to Dormieux et al. (Dormieux et al. 2006, 2007), The

effective energy writing for the case of n-layered CSA with imperfect interfaces is

$$\begin{aligned} \mathcal{W}_{hom}(\underline{\underline{E}}, p_l) &= \sum \frac{\Omega_i}{\Omega} \mathcal{W}_i(\underline{\underline{\varepsilon}}_i, p_l) + \sum \frac{\Gamma_i}{\Omega} \mathcal{W}_i^\Gamma([i, p_l]) \\ &\quad \sum \frac{\Omega_i}{\Omega} \mathcal{W}_i(\underline{\underline{\varepsilon}}_i, p_l) + \sum \frac{3}{r_i} \frac{\Omega_i}{\Omega} \mathcal{W}_i^\Gamma([i, p_l]) \\ &\quad \sum f_i \mathcal{W}_i(\underline{\underline{\varepsilon}}_i, p_l) + \sum \frac{3f_i}{r_i} \mathcal{W}_i^\Gamma([i, p_l]) \end{aligned} \quad (23)$$

where $\mathcal{W}_i$ is energy of the phase i (with volume fraction $f_i = \Omega_i/\Omega$); $\mathcal{W}_i^\Gamma$ is interface energy at the interface Γ_i between phase (i) and ($i+1$); $[\xi^i]$ is the displacement jump across the interface Γ_i , in which the discontinuity of radial displacement is associated with the fluid pressure p_l by Eq. (12). The relation between the local stress and strain field within the phase i reads

$$\underline{\underline{\sigma}} = \mathbb{C}_i : \underline{\underline{\varepsilon}} - b_i p_l \underline{\underline{i}} \text{ in } \Omega_i \quad (24)$$

and the energy of phase i is obtained by relation with Biot tensor $\underline{\underline{b}}$ and solid Biot modulus N_i

$$\mathcal{W}_i \left(\begin{bmatrix} \underline{\underline{\varepsilon}} \\ p_l \end{bmatrix} \right) = \frac{1}{2} \underline{\underline{\varepsilon}} : \mathbb{C}_i : \underline{\underline{\varepsilon}} - p_l \underline{\underline{b}}_{-i} : \underline{\underline{\varepsilon}} - \frac{1}{2} \frac{p_l^2}{N_i} \quad (25)$$

The second term in Eq. (23) is the contribution from the interface Γ_i . Modeling the interface with zero thickness neglects its porosity and volume, resulting in a negligible fluid storage capacity within the discontinuity. The interface energy can be written in the following form

$$\mathcal{W}_i^\Gamma([\xi_r^i], p_l) = \frac{1}{2} [\xi_r^i] D_r^i [\xi_r^i] - p_l D_r^i \omega_r^i [\xi_r^i] \quad (26)$$

where σ_{rr}^i and $[\xi_r^i]$ are respectively the stress and the displacement jump across the interface Γ_i , which are given by Eq. (12)

$$\begin{cases} \sigma_{rr}^i = D_r^i [\xi_r^i] - D_r^i \omega_r^i p_l \text{ with } i = (1, n) \\ [\xi_r^i] = \xi_r^{i+1} - \xi_r^i \end{cases} \quad (27)$$

where

$$\xi_r^i = H_i r + \frac{L_i}{r^2} \quad (28)$$

Deriving $\mathcal{W}_{hom}$, $\mathcal{W}_i$ and $\mathcal{W}_i^F$ in the relations (22), (25) and (26) with respect to p_l yield

$$\begin{aligned} \frac{\partial \mathcal{W}_{hom}}{\partial p_l} &= -\frac{p_l}{N_{hom}} \\ \frac{\partial \mathcal{W}_i}{\partial p_l} &= -\frac{b_i}{-i} : \varepsilon - \frac{p_l}{N_i} \\ \frac{\partial \mathcal{W}_i^F}{\partial p_l} &= -D_r^i \omega_r^i [\xi_r^i] \end{aligned} \quad (29)$$

where the strain is deduced from the non-zero displacement (see Eq. (10))

$$\begin{cases} \xi_r^i = H_i r + \frac{L_i}{r^2} \\ \varepsilon = \frac{1}{2} \left(\text{grad} \xi^i + \text{grad}^T \xi^i \right) \\ -i \end{cases} \quad (30)$$

and the non-zero components

$$\begin{aligned} \varepsilon_{rr}^i(r) &= H_i - \frac{2L_i}{r^3} \\ \varepsilon_{\theta\theta}^i(r) &= \varepsilon_{\varphi\varphi}^i(r) = H_i + \frac{L_i}{r^3} \end{aligned} \quad (31)$$

where H_i is determined from sub-problem (P'') in the subsection 2.1. Using $\frac{b_i}{-i} = b_i \delta$ in which b_i is coefficient Biot, Eq. (29) becomes

$$\begin{aligned} \frac{\partial \mathcal{W}_{hom}}{\partial p_l} &= -\frac{p_l}{N_{hom}} \\ \frac{\partial \mathcal{W}_i}{\partial p_l} &= -3b_i H_i - \frac{p_l}{N_i} \\ \frac{\partial \mathcal{W}_i^F}{\partial p_l} &= -D_r^i \omega_r^i [\xi_r^i] \end{aligned} \quad (32)$$

This equation results in the relationship between the effective solid Biot modulus of n -phase CSA and the poroelastic properties of its constituents.

$$\frac{1}{N_{n+1}} = \frac{1}{N_{hom}} = \sum f_i \left(\frac{1}{N_i} + \frac{3b_i H_i}{p_l} \right) + \sum \frac{3f_i}{r_i} \left(\frac{D_r^i \omega_r^i [\xi_r^i]}{p_l} \right) \quad (33)$$

where coefficients (H_i, L_i) and the displacement jumps $[\xi_r^i]$ are determined from subproblem (P'') in subsection 2.1.

2 Application to Oolitic Limestones

2.1 Microstructure of Materials

The oolitic limestone is a mono crystalline rock composed of 98% calcite. The microphotograph of this rock shows that the studied rock is constituted of quasi-spherical ooids (oolite) cemented by a sparitic calcite (see Fig. 1a). Different observation techniques including X-ray diffraction (XRD), optical microscopy (OM) and scanning electron microscopy coupled with an energy dispersive spectrometer (SEM-EDS) have evidenced that the oolite is similar to a granular material with a microporosity between fine grains (micrite) (see Fig. 1c). The cement linking the oolites is composed of sparite with a grain size bigger than the micrite grains within the oolites. Both sparite and micrite are entirely composed of calcite ((Fabre and Gustkiewicz 1997; Gourri 1991; Lion 2004; Lion et al. 2004; Trieu et al. 2020; Nguyen 2010; Hart and Wang 1995)). As shown in Fig. 1b, the porosity of the rock is mostly located at the interface between the oolite and the surrounding matrix, which is often considered as a highly porous transition zone (ITZ) (Grgic 2011; Auvray 2010). This outer layer has been often considered as a thin, flexible interphase (for instance, see (Nguyen 2010; Nguyen et al. 2011; Giraud et al. 2012)). However, it can be also represented as an imperfect interface.

2.2 A Two-Step Homogenization Approach

To estimate the poroelastic properties of oolitic rocks, these materials are considered as an assemblage of spherical ooids coated by sparite matrix with an imperfect bonding between them. From a micromechanical point of view, it is a 2-layered concentric spherical composite material with an imperfect interface, i.e. the 3-layered CSA model is used for oolitic rocks. Each phase is made from calcite (solid matrix) and pore (see Fig. 1) and modelled as a porous isotropic medium. This assumption is consistent with classical poromechanics approaches for carbonate rocks and is justified by the relatively homogeneous mineral composition (calcite) at the microscale. Therefore, a two-step homogenization approach (micro-meso and meso-macro) is proposed to estimate the poroelastic properties of studied materials. In the following, pore and calcite (micrite, sparite) are referred as the microscale; while ooid, interphase and cement matrix are considered as the mesoscale; and finally, a REV of the limestone is envisaged as the macroscale. Figure 2 shows the principal scheme of the proposed two-step micromechanical model. At the first step (step I), the oolite and the matrix are homogenized as the porous material polycrystal with spherical pores, which is performed by using the iterative self-consistent scheme (Hill 1965; Berryman 1980; Zouari

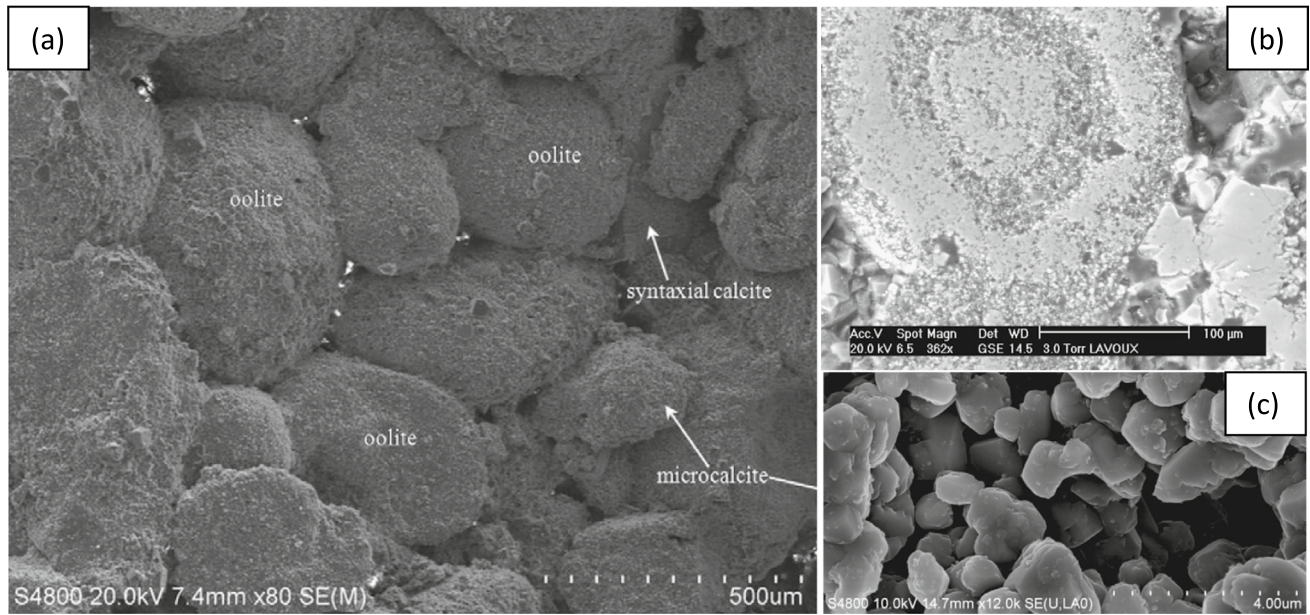


Fig. 1 Oolitic Lavoux limestone (Grgic 2011; Auvray 2010; Sterpenich et al. 2009): **a** Low magnification SEM (Scanning Electron Microscopy), **b** concentric layers of oolites, **c** microstructure of oolites: micritic calcite grains and micropores

et al. 2008). At the second step, the effective poroelastic properties of the rock are computed in the framework of 3-phase CSA model with imperfect interface from properties of mesoscopic components, including oolite (o), cement matrix (m) and of imperfect interface. The resulting effective properties correspond to a macroscopic (engineering-scale) description of the material, suitable for use in larger-scale models.

The total porosity is decomposed into two parts, oolite porosity f_{op} and matrix porosity f_{mp} , which are assumed to be connected to each other

$$\begin{aligned} \Omega &= \Omega_o + \Omega_m, \quad f_o = \frac{\Omega_o}{\Omega}, f_m = \frac{\Omega_m}{\Omega}, \quad f_{op}^I = \frac{\Omega_{op}}{\Omega_o} = \frac{f_{op}}{f_o} \\ \Omega_p &= \Omega_{op} + \Omega_{mp}, \quad f_{op} = \frac{\Omega_{op}}{\Omega_p} = \eta_o f_p, \quad f_{mp}^I = \frac{\Omega_{mp}}{\Omega_m} = \frac{f_{mp}}{f_m} \\ \Omega_{op} &= \eta_o \Omega_p, \quad f_{mp} = \frac{\Omega_{mp}}{\Omega_p} = (1 - \eta_o) f_p, \quad f_o + f_m = 1 \end{aligned} \quad (34)$$

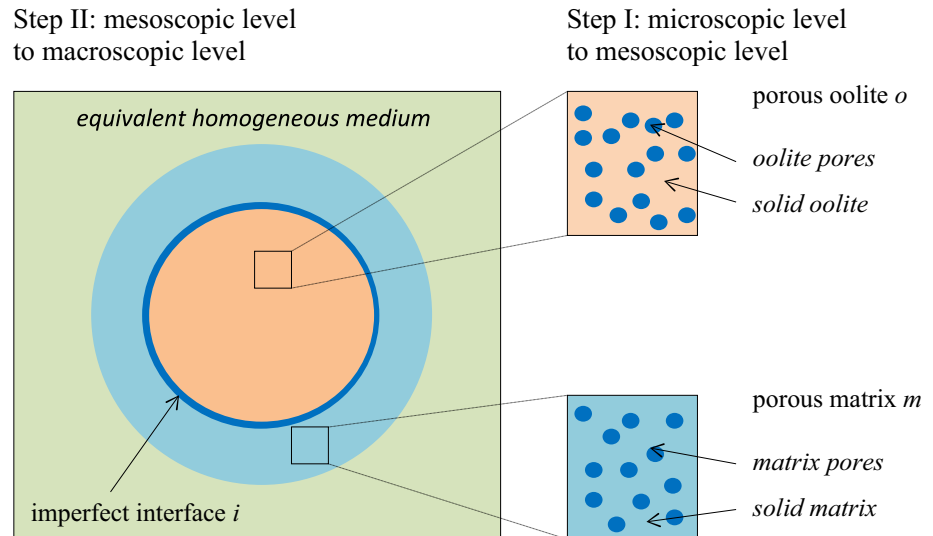
where f_{op}^I and f_{mp}^I are respectively the oolite and matrix porosities at step I and η_o is relative porosity of oolites. They must satisfy the following conditions

$$f_{op}^I = \frac{\eta_o f_p}{f_o} < 1, f_{mp}^I = \frac{(1 - \eta_o) f_p}{f_m} < 1 \quad (35)$$

The solid particles constituting oolites (o) and surrounding matrix (m) are both assumed to be isotropic and made of the pure calcite. Elastic properties of solid grain of phase j ($j = o, m$) reads

$$\begin{aligned} \mathbb{C}_j &= 3k_j \mathbb{J} + 2\mu_j \mathbb{K} \\ k_j &= \chi_j k_{ca}, \mu_j = \chi_j \mu_{ca}, k_{ca} \approx 76.0 \text{ GPa}, \mu_{ca} \approx 36.8 \text{ GPa} \end{aligned} \quad (36)$$

where the coefficient χ_j are unknown and can be directly deduced from the solid matrix bulk modulus of the rock (see subsection 3.3); $\mathbb{C}_j$ is the elastic stiffness tensor; k_j, μ_j are bulk and shear moduli of solid component of phase j . The coefficients χ_j are introduced as effective scaling factors to account for the reduction of the elastic properties of calcite within each constituent (oolite and matrix). They reflect the overall degradation of stiffness due to microstructural features at a lower scale (e.g., defects, microcracks, or unresolved heterogeneities). The same scaling factor is applied to both the bulk and shear moduli, which implies a preserved Poisson's ratio. This simplifying assumption was adopted to limit the number of independent parameters. It should be noticed that only experimental characterization performed thanks to micro or nano-indentation tests would allow to determine relative stiffness parameters of matrix (m) and oolite (o). For the particular case of oolitic limestones, the assumption $\chi_m < \chi_o < 1$ is made according to microstructural information (Fabre and Gustkiewicz 1997; Gourri 1991; Lion 2004; Lion et al. 2004; Trieu et al. 2020; Nguyen 2010; Hart and Wang 1995). However, it may be different from other oolitic limestones, for instance $\chi_o < \chi_m < 1$ in Ghabezloo et al. (Ghabezloo et al. 2009).

Fig. 2 A two-step model for porous oolitic limestones

2.2.1 Step I: Poroelastic Properties of the Oolite and the Matrix

It is well known that the self-consistent scheme (SC) may become inaccurate or even degenerate for porosity approaching 0.5, while the Mori–Tanaka scheme (MT) is generally better suited to porous media with low porosity. By contrast, differential-type schemes (like differential self-consistent and iterative self-consistent schemes) are known to provide more realistic estimates for highly porous materials. Nguyen et al. (Nguyen et al. 2011) compared different homogenization approaches for the case of oolitic rocks, including Mori–Tanaka (MT), Hashin Shtrikman upper bound (HS+), Self-consistent (SC), Generalized self-consistent (GSC), Differential Self-consistent (DSC) and Iterative Self-consistent (ISC) (with different iteration n). The authors concluded that ISC and DSC seem the most relevant for the material considered. Therefore, the iterative self-consistent scheme is chosen to the first step to homogenize the oolites and matrix containing microporosities. This step is referred as the transition from the microscopic to the mesoscopic scale. Micro-pore is assumed to be spherical. The iterative self-consistent scheme principle proposed by Zouari et al. (Zouari et al. 2008) is inspired from the differential self-consistent scheme (see Zouari et al. 2008; Smaoui 2007) for more details).

To reach an effective porous medium, n successive intermediate media obtained by addition of pore fractions Δf_{jp}^I are elaborated, until the final porosity f_{jp}^I is reached. At a step (i), the constituents of the intermediate medium are derived from the precedent step ($i-1$) such as

$$\frac{k_j^{(i)}}{k_j^{(i-1)}} = \frac{1-f_{jp}^{I(i-1)}}{1+\alpha_j^{(i)}(k_j^{(i-1)}-k_j^{(i)})/k_j^{(i)}}, \quad \frac{\mu_j^{(i)}}{\mu_j^{(i-1)}} = \frac{1-f_{jp}^{I(i-1)}}{1+\beta_j^{(i)}(\mu_j^{(i-1)}-\mu_j^{(i)})/\mu_j^{(i)}} \quad (37)$$

$$\alpha_j^{(i)} = \frac{3k_j^{(i)}}{3k_j^{(i)}+4\mu_j^{(i)}}, \quad \beta_j^{(i)} = \frac{2(k_j^{(i)}+2\mu_j^{(i)})}{5(3k_j^{(i)}+4\mu_j^{(i)})}$$

with

$$k_j^{(0)} = k_j, \mu_j^{(0)} = \mu_j, f_{jp}^{I(0)} = 0$$

$$\Delta f_{jp}^I = \frac{f_{jp}^I}{n}, f_{jp}^{I(i)} = \frac{\Delta f_{jp}^I}{1-f_{jp}^I + i\Delta f_{jp}^I} \quad (38)$$

The iterative process (70) and (71) is performed from $i=1$ to the final state $i=n$ and so that $k_j^I = k_j^{(n)}, \mu_j^I = \mu_j^{(n)}$. In the case of isotropic homogeneous solid, the coefficient Biot and the solid Biot modulus read as

$$b_j^I = 1 - \frac{k_j^I}{k_j}, \quad \frac{1}{N_j^I} = \frac{b_j^I - f_{jp}^I}{k_j} \quad (39)$$

The result of this first step is two isotropic homogenized linear poroelastic materials: porous oolite and porous matrix, with isotropic linear poroelastic properties denoted $k_j^I, \mu_j^I, b_j^I, N_j^I, (j=0, m)$.

2.2.2 Step II: Poroelastic Properties of Limestone

The oolites and surrounding matrix are homogenized at the second step in the framework of Hashin composite sphere assemblage (CSA) models: grains (oolites) are assumed to be spherical and the surrounding matrix is modelled by a spherical shell coating the grains. The interfacial transition zone between oolite and matrix is modelled as an imperfect interface. The three phase composite sphere with an imperfect interface is used as a particular case of the $n+1$ -phase

model that its solution is well known in Sect. 2 to obtain effective properties of the porous composite.

For the 3-phase CSA model ($n=2$), the effective bulk modulus is easily deduced thanks to Eq. (22) (subsection 2.1) such as

$$k_{hom}^{II} = \frac{4f_o k_o^I \mu_o^I \mu_m^I + k_m^I [4(1-f_o) \mu_o^I \mu_m^I + 3k_o^I (4(1-f_o) \mu_m^I p + \mu_o^I)]}{4\mu_o^I \mu_m^I + 3k_o^I (4\mu_m^I p + \mu_o^I) + 3f_o [k_m^I (3k_o^I p + \mu_o^I) - k_o^I \mu_o^I]} \quad (40)$$

Thanks to Eq. (42) (subsection 2.2), the effective shear modulus is the positive root of the following second order equation

$$\mathcal{A} \left(\frac{\mu_{hom}^{II}}{\mu_m^I} \right)^2 + \mathcal{B} \left(\frac{\mu_{hom}^{II}}{\mu_m^I} \right) + \mathcal{C} = 0 \quad (41)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are determined by Eqs. (43)-(45) with

$$\begin{aligned} Z_{14} &= 7f_o^{\frac{5}{3}} (4v_m^I - 5) \left[\eta_7 (\mu_o^I)^2 - 4\eta_1 (\mu_m^I)^2 + \eta_2 \mu_o^I \mu_m^I \right] \\ Z_{21} &= 4v_m^I \left[4\eta_1 (5v_m^I - 7) (\mu_m^I)^2 + 2\eta_7 (5v_m^I - 4) (\mu_o^I)^2 + \eta_3 \mu_o^I \mu_m^I \right] \\ Z_{23} &= -18f_o^{\frac{5}{3}} v_m^I \left[\eta_7 (\mu_o^I)^2 - 4\eta_1 (\mu_m^I)^2 + \eta_2 \mu_o^I \mu_m^I \right] \\ Z_{24} &= -10f_o v_m^I (4v_m^I - 5) \left[\eta_7 (\mu_o^I)^2 - 4\eta_1 (\mu_m^I)^2 + \eta_4 \mu_o^I \mu_m^I \right] \\ Z_{31} &= -5f_o^{\frac{7}{3}} \left[2\eta_1 (\mu_m^I)^2 \left((v_m^I)^2 - 7 \right) + \eta_7 (\mu_o^I)^2 \left(8(v_m^I)^2 - 12v_m^I + 7 \right) + \eta_5 \mu_o^I \mu_m^I \right] \\ Z_{34} &= -f_o^{\frac{10}{3}} (4v_m^I - 5) \left[\eta_1 (\mu_m^I)^2 (5v_m^I + 7) - \eta_7 (\mu_o^I)^2 (10v_m^I - 7) + \eta_6 \mu_o^I \mu_m^I \right] \end{aligned} \quad (42)$$

$$\begin{aligned} \eta_1 &= 7(2p+1)(2q+1) + 2v_o^I [(10q-1)p - 4q - 5] \\ \eta_2 &= 2\eta_7 (2v_m^I - 1)q - 4\eta_7 (v_m^I - 2)p - 45v_o^I + 21 \\ \eta_3 &= 2\eta_7 (19v_m^I - 17)q + 4\eta_7 (13v_m^I - 11)p + 315v_m^I - 15v_o^I (25v_m^I - 19) - 273 \\ \eta_4 &= 7(2q+4p+3) + 5v_o^I (2q+4p-9) \\ \eta_5 &= 2\eta_7 \left[4(v_m^I)^2 + 6v_m^I - 7 \right] q + 4\eta_7 \left[2(v_m^I)^2 - 6v_m^I + 7 \right] p - 4(v_m^I)^2 (5v_o^I - 14) - 6v_m^I (11v_o^I + 7) - 49(v_o^I - 1) \\ \eta_6 &= v_m^I [7(8q+2p+5) + 10v_o^I (4q+p+10)] - 35v_o^I (2q+2p-1) - 98(q+p+1) \\ \eta_7 &= 5v_o^I + 7 \end{aligned} \quad (43)$$

where p, q are dimensionless interface elastic parameters defined by Eq. (47). Hashin (Hashin 1991) shows that a thin, flexible interphase layer is equivalent to an imperfect interface and the interface parameters can be determined from interphase properties and thickness. The effect of degrading interface bond, from perfect bond to non-bonded, on properties has been shown in Hashin (Hashin 1991). Moreover, as seen in Eq. (47), the cases when $p=0$ and/or $q=0$ ($D_r \rightarrow \infty$ and/or $D_\theta \rightarrow \infty$) correspond to perfect normal and/or shear bond. In this case, we can recall the classical equation of the three-phase model with perfect interface (see Christensen and Lo (Christensen and Lo 1979) and Hervé and Zaoui (Hervé and Zaoui 1993)).

Equations (20) and (33) result in the expressions of the effective coefficient Biot and the solid Biot modulus for three phase ($n=2$)

$$b_{hom}^{II} = \frac{f_o \mu_o^I (3k_m^I + 4\mu_m^I) (b_o^I + 3s) + b_m^I (1-f_o) [4\mu_o^I \mu_m^I + 3k_o^I (4\mu_m^I p + \mu_o^I)]}{4\mu_o^I \mu_m^I + 3k_o^I (4\mu_m^I p + \mu_o^I) + 3f_o [k_m^I (3k_o^I p + \mu_o^I) - k_o^I \mu_o^I]} \quad (44)$$

$$\frac{1}{N_{hom}^{II}} = \frac{f_o}{N_o^I} + \frac{1-f_o}{N_m^I} + 3f_o \frac{\mathfrak{B}}{\mathfrak{D}} \quad (45)$$

where

$$\begin{aligned} \mathfrak{B} &= \frac{(k_o^I)^2 p^2 [3(1-f_o) k_o^I (b_m^I)^2 + (3f_o k_m^I + 4\mu_m^I) (b_o^I)^2] - k_o^I \mu_o^I p [(f_o - 1) k_o^I (b_m^I)^2 + 2(1-f_o) k_o^I b_m^I (b_o^I + 3s)] + k_o^I \mu_o^I p [(f_o - 1) k_o^I (b_o^I)^2 + 2(3f_o k_m^I + 4\mu_m^I) b_o^I s] + (\mu_o^I)^2 s^2 [3(1-f_o) k_o^I + (3f_o k_m^I + 4\mu_m^I)]}{\mathfrak{D} = (k_o^I)^2 p [4\mu_o^I \mu_m^I + 3k_o^I (4\mu_m^I p + \mu_o^I)] + 3f_o (k_m^I (3k_o^I p + \mu_o^I) - k_o^I \mu_o^I) - k_o^I \mu_o^I} \end{aligned} \quad (46)$$

and

$$s = \frac{\omega_r k_o^I}{r_o}, p = \frac{\mu_o^I}{r_o D_r}, q = \frac{\mu_o^I}{r_o D_\theta} \quad (47)$$

with s is a dimensionless parameter, r_o is the radius of oolite and ω_r is the interface poro-elastic parameter in the normal direction. Similar of thermo-elastic interface parameters of sphere composite with imperfect interface has been presented in Duan et al. (Duan et al. 2007a). For the case of poro-elastic (or thermo-elastic) interface, we can find the equivalence between a thin, flexible interphase (thickness t , elastic properties k_i, μ_i and coefficient Biot b_i) and a linear spring model by the following relation.

$$\begin{aligned}\omega_r &= \frac{3b_i}{3k_i+4\mu_i}t \\ D_r &= \frac{3k_i+4\mu_i}{3t} \\ D_\theta &= \frac{3t}{t}\end{aligned}\quad (48)$$

It can be seen from (47) that in the case when $p=0$ and $s=0$ ($D_r \rightarrow \infty$ and $\omega_r = 0$), the expression of the coefficient Biot for the three phase model with perfect interface is identical to the formulation obtained by Bary and Béjaoui (Bary and Béjaoui 2006). In this study, the high-porosity transition zone between the oolites and the matrix is modeled as an imperfect interface, whose properties are defined as being proportional to the porosity.

$$p = \alpha f_p, q = \beta f_p, s = \gamma f_p \quad (49)$$

In summary, the proposed two-step homogenization model (i.e., ISC for the first step and the 3-phase CSA model ($n=2$) with the imperfect interface for the second step) considered for the oolitic rock includes following parameters:

- f_o or $f_m = 1 - f_o$ the volume fraction of oolite and matrix given in Eqs. (34) and (35).
- f_{op}, f_{mp} the porosity of oolite and matrix or the total porosity f_{op} and the relative porosity of oolites η_o defined in Eqs. (34) and (35).
- $k_o^s, \mu_o^s, k_m^s, \mu_m^s$ the solid skeleton bulk and shear moduli of oolite and matrix introduced in Eq. (36). They are represented by two intermediate dimensionless parameters χ_o, χ_m .
- Imperfect interface parameters including D_r the normal stiffness, D_θ the tangent stiffness (see Eq. (3)), ω_r the interface poro-elastic parameters in the normal direction representing the discontinuity of radial displacement only caused by the fluid pressure (see Eq. (12)), radius of ooid r_o (see Eq. (47)). These parameters are dimensionless via 3 parameters p, q, s (see Eq. (47)) or α, β, γ (see Eq. (49)).

2.3 Results and Discussion

A literature review gathers experimental data of poroelastic properties of various oolitic limestones in Table 1 (see also Bemmer et al. (Bemer et al. 2004)). Not all the parameters required by the proposed model are not available for all these considered rock. Fortunately, key microstructure parameters, namely the volume fraction of ooids f_o and its relative porosity η_o are available for Lavoux limestone from the literature (see (Nguyen 2010; Grgic 2011; Auvray 2010)).

$$f_o = 0.7, 0 \leq f_p \leq 0.3, \eta_o = 0.6 \quad (50)$$

These parameters are therefore adopted as reference values for all considered limestones, in order to maintain

a consistent basis for comparison. The different limestones are then used to assess the ability of the model to reproduce general trends as a function of porosity.

Condition (35) imposes that

$$\begin{aligned}0 \leq f_{op} \leq 0.18, 0 \leq f_{mp} \leq 0.12 \\ 0 \leq f_{op}^I \leq \frac{0.18}{0.7} \approx 0.257, 0 \leq f_{mp}^I \leq \frac{0.12}{1-0.7} = 0.4\end{aligned}\quad (51)$$

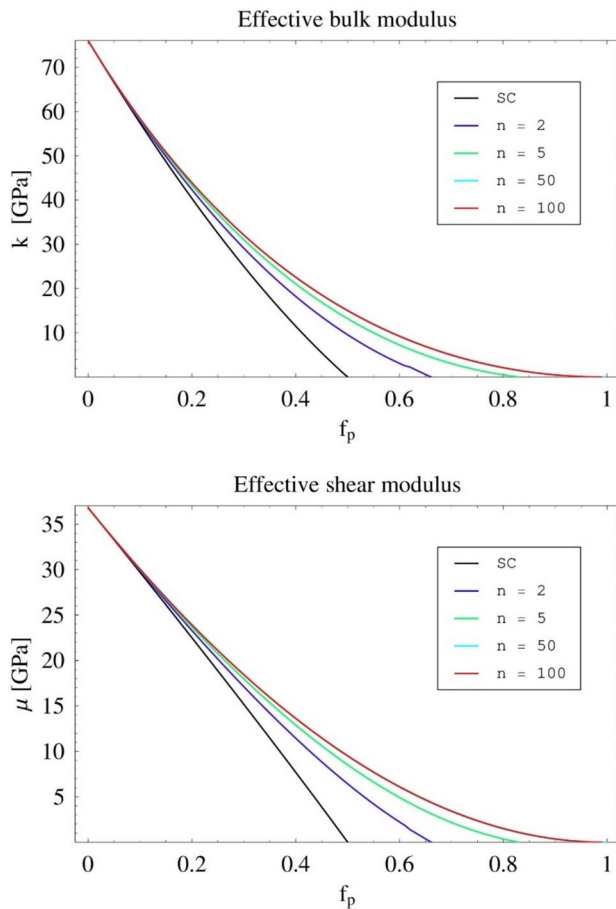
The other parameters ($\chi_o, \chi_m, \alpha, \beta$ and γ) can be determined from the effective linear poroelastic experimental data.

As a reminder, the iterative self-consistent scheme (ISC) has been chosen at the first homogenization step to compute the effective properties of each phase (oolite and surrounding matrix). A comparison of this homogenization scheme is presented in Fig. 3 for different values of increments ($n=1, 2, 5, 10, 100$). The case $n=1$ corresponds to the classical self-consistent scheme (SC). It should be noticed that the self-consistent results in zero bulk and shear moduli beyond the porosity $f_p = 0.5$, which is often interpreted as a percolation threshold of the pores (Vu et al. 2013, 2014, 2018, 2017; Pouya and Vu 2012a, 2012b; Nguyen et al. 2020). On the other hand, ISC scheme may be used for high values of porosity ($f_p > 0.5$) (see Smaoui et al. (Smaoui et al. 2006)). For the porous material of pure calcite with spherical pores, ISC estimate converges for a sufficiently large number of increments ($n=100$ in Fig. 3). Therefore, the number of increments $n=100$ will be chosen for the ISC scheme at the first step of homogenization.

At the second homogenization step, the solid matrix bulk modulus k_s^H is assumed to be constant for all the considered limestones. This assumption corresponds to the purpose of proposing a unique set of model parameters applicable to all the considered oolitic limestones. The bulk modulus of solid grains of the oolitic rocks listed in Table 1, as a function of their porosity is presented in Fig. 4. The parametric study presented in this figure suggests the values $\chi_o = 1$ and $\chi_m = 0.5$ (Eq. (36)) corresponding to the mean value of data ($k_s^H \approx 60 \text{ GPa}$). Using chosen parameters ($f_o = 0.7, \eta_o = 0.6$) and two fitting parameters ($\chi_o = 1, \chi_m = 0.5$), a sensitivity analysis for the imperfect interface parameters (α, β and γ or p, q and s) is presented in Fig. 5. It can be observed that the model reproduces globally the experimental trends. It also highlights that the interface parameters control the effective macroscopic properties of oolitic limestones. The case $\alpha = 0, \beta = 0$ and $\gamma = 0$ corresponding to the perfect bond interface results in the upper bound of effective macroscopic properties. An increase in degrading interface bond, represented by an increase of the interface elastic parameters (α, β and γ or p, q and s), leads to a decrease in the effective elastic properties. This sensitivity study also gives a set of parameters, which can be used for all the considered oolitic rock as shown in Fig. 6.

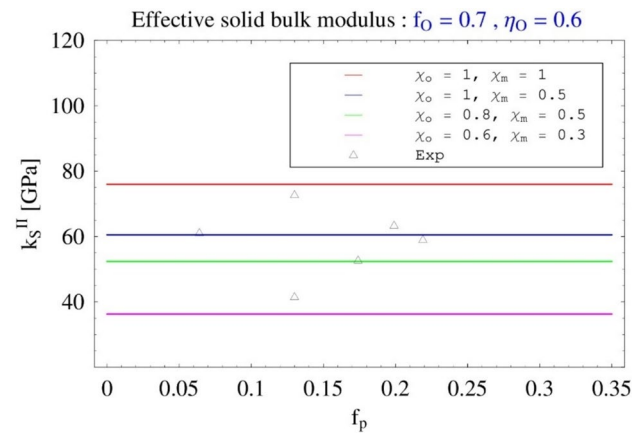
Table 1 Experimental data of various oolitic limestones

Limestone	f_p	k^{exp} (GPa)	k_s^{exp} (GPa)	b^{exp}
Villeperdue (Fabre and Gustkiewicz 1997; Gourri 1991)	0.064	36	61	0.41
Tonnerre (Fabre and Gustkiewicz 1997; Gourri 1991; Lion 2004)	0.13	19.3	41.4	0.53
Salem (Hart and Wang 1995)	0.13	21.78	72.6	0.7
Lavoux (Fabre and Gustkiewicz 1997; Gourri 1991; Lion 2004)	0.219	13.8	58.9	0.77
Chauvigny (Fabre and Gustkiewicz 1997; Gourri 1991; Lion 2004)	0.174	16.3	52.6	0.69
Anstrude (Gourri 1991; Lion 2004)	0.199	15.5	63.25	0.82

**Fig. 3** Iterative self-consistent schemes for different values of increments n

As mentioned at the end of Sect. 3.2, the definition of model parameters were distributed throughout the paper. To enhance the model's readability and reproducibility, all the parameters and their sources are summarized in Table 2.

The effective properties derived in this work via two-step homogenization approach corresponds to macroscopic description and are used as data input for large scale simulation where the material can be considered homogeneous (Seyedi et al. 2015; Vu et al. 2025; Plua et al. 2021a, 2021b).

**Fig. 4** Solid matrix bulk modulus: effect of the parameters χ_o and χ_m

The model does not explicitly account for multiscale coupling or size-dependent effects.

It is worth noting that while the model is illustrated using oolitic limestones, which are predominantly composed of calcite, the proposed framework is not restricted to monomineral rocks. The formulation is based on homogenization of porous phases and can be applied to polycrystalline geomaterials, provided that each phase can be represented by effective isotropic poroelastic properties at the considered scale. Moreover, the proposed model assumes an isotropic behavior of each phase, which is consistent isotropic microstructure of the considered oolitic limestones and the unavailable data about the anisotropy of these considered rocks. For more general geological formations exhibiting anisotropy (e.g., layered or oriented microstructures), the model represents an equivalent isotropic approximation. The anisotropy can be considered by considering the n -layered ellipsoidal assemblage.

3 Conclusion

This paper presents the analytical solution of effective poroelastic properties of an n -layered concentric sphere composite with an imperfect interface between two consecutive layers in the framework of ' $n+1$ '-phase CSA model. To do

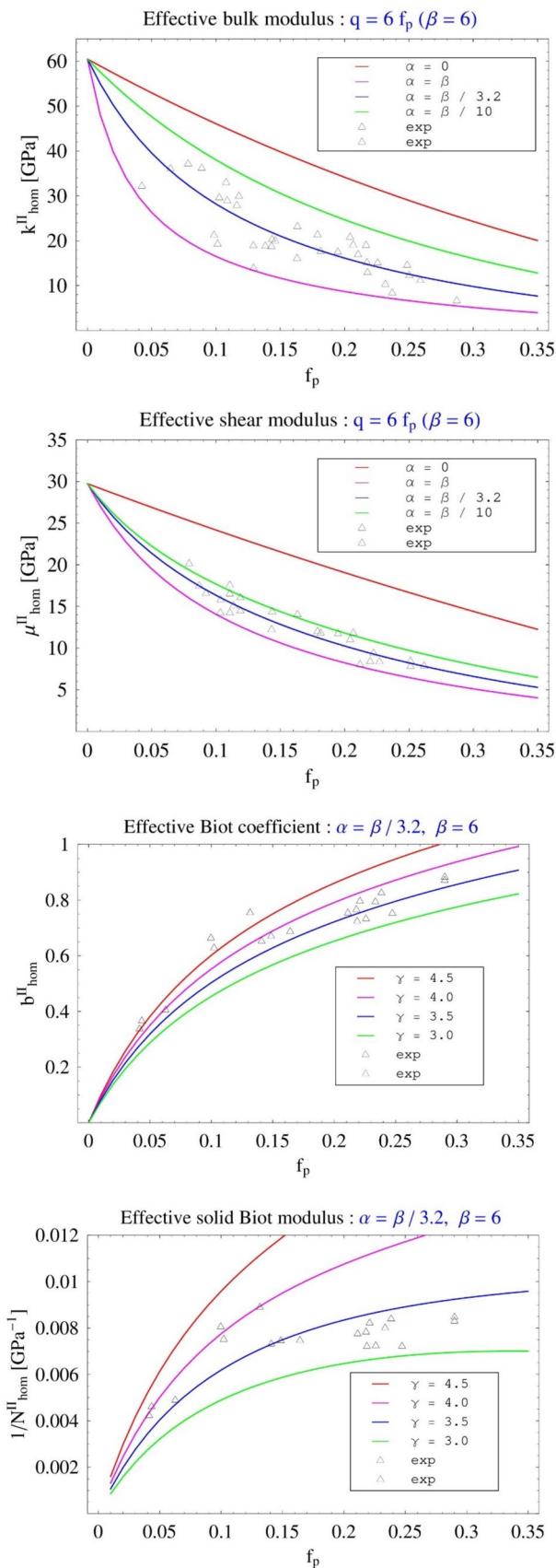


Fig. 5 A sensitivity analysis to poroelastic parameters

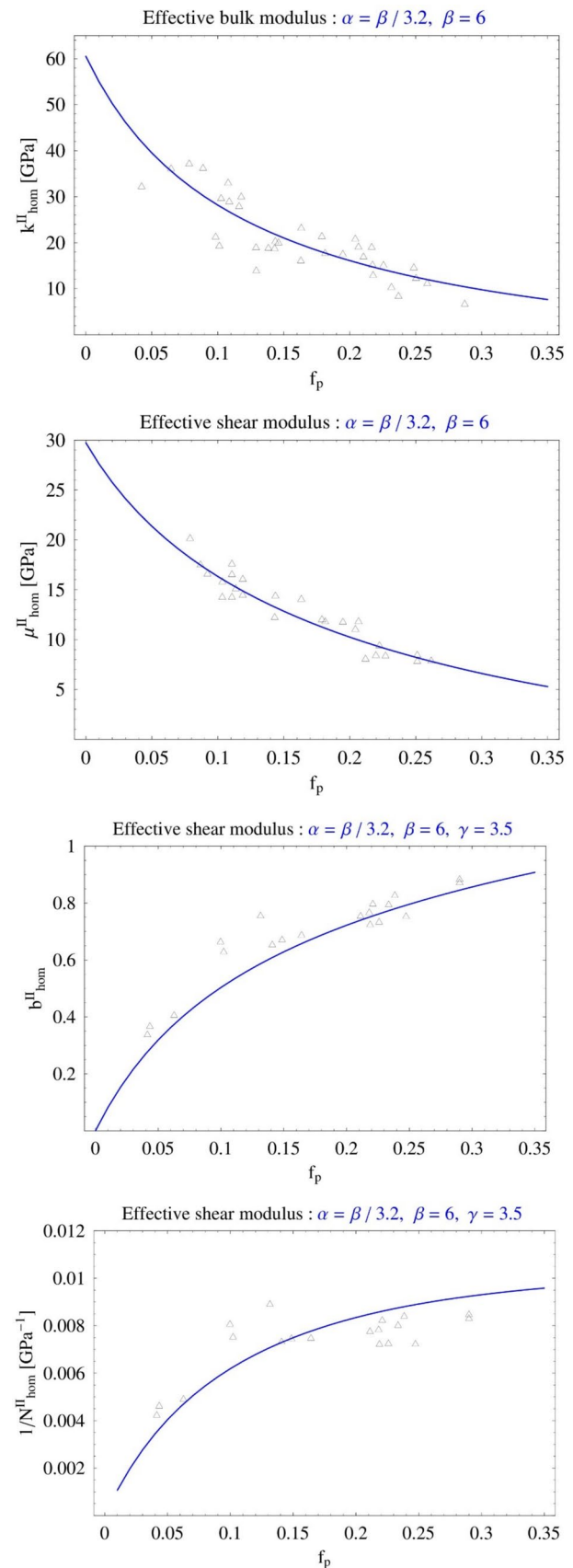


Fig. 6 Effective poroelastic properties of oolitic limestone

Table 2 Model parameters and their origin

Parameters	Alternative representation	Meaning	Source/calibration
f_o	–	Volume fraction of ooids	Literature (data from Lavoux limestone (see (Grgic 2011; Auvray 2010))
f_p	–	Total porosity	Literature (see Table 1)
η_o	Porosity of oolites $f_{op} = \eta_o f_p$	Relative porosity of oolites	Literature (data from Lavoux limestone (see (Grgic 2011; Auvray 2010))
k_o^s, μ_o^s	Coefficient χ_o relating k_o^s, μ_o^s to the bulk and shear moduli of pure calcite $k_{ca} \approx 76.0 \text{ GPa}$ and $\mu_{ca} \approx 36.8 \text{ GPa}$ $k_o^s = \chi_o k_{ca}$ $\mu_o^s = \chi_o \mu_{ca}$	Bulk and shear moduli of solid skeleton of oolites	Fitting the experiment data in Table 1 (see Fig. 4)
k_m^s, μ_m^s	Coefficient χ_m $k_m^s = \chi_m k_{ca}$ $\mu_m^s = \chi_m \mu_{ca}$	Bulk and shear moduli of solid skeleton of cement matrix	Fitting the experiment data in Table 1 (see Fig. 4)
$D_r, D_\theta, \omega_r, r_0$	p, q, s (see Eq. (79)) or α, β, γ (see Eq. (81))	LMS model parameters for the imperfect interface	Fitting the experiment data in Table 1 (see Figs. 5 and 6)

so, this study extends the mathematical derivation from the thermoelastic derivation shown by Hashin (Hashin 1991) for only three phase sphere composites with imperfect interface to a poroelastic solution for the multi-coated sphere model. Linear Spring Model is used to describe the interface behavior, in which the displacement is discontinuous. The relation between displacement discontinuity across the interface and stress is linear.

According to the generalized self-consistent scheme or the three-phase CSA model proposed by Christensen and Lo (Christensen and Lo 1979), an n-layered porous sphere composite is assumed to be embedded in the (n + 1) layer with equivalent poroelastic properties. The (n + 1)-layered composite is subjected to a macroscopic strain at infinity and a pore pressure field within the whole domain. This problem is split into two subproblem. The first one corresponds to the drained condition (i.e. variation in pore pressure is nil) under the far field condition of the macroscopic strain. In the second one, the whole porous medium is subjected to a pore pressure under the volume constant condition. The second one allows to determine the Biot coefficient and the solid Biot modulus. The main novelty of this study consists in providing a direct derivation of closed-form solutions of the Biot coefficient, and solid the Biot modulus for n-layered spherical composites with an imperfect interface.

The present ‘n + 1’-phase CSA model is applied to study the effective poroelastic properties of various oolitic limestones. The microstructure of these rocks exhibits an assemblage of nearly spherical ooids coated by a sparite cement matrix. Moreover, oolitic grains and the sparite matrix are also porous medium, which is constituted by solid grain and micropores. The whole porosity is assumed to be connected throughout the porous composite. A highly porous zone is present at the border of the oolite, which is considered in this study by an imperfect interface. A two-step homogenization

approach (micro-meso and meso-macro transitions) is proposed to compute the poroelastic properties of oolitic rocks. In such the micro–meso and meso–macro transitions, referring to the hierarchical homogenization procedure adopted, pore and calcite (micrite, sparite) are referred as the micro-scale; while ooid and cement matrix are considered as the mesoscale; and finally, a REV of the limestone is envisaged as the macroscale. The first step aims to estimate the properties of the oolite and the cement matrix by using the iterative self-consistent scheme. The second step consists in deriving the effective poroelastic properties of the rock by using the three phase CSA model with imperfect interface from properties of mesoscopic components, including oolite (o), cement matrix (m) and imperfect interface.

The comparison with experimental data shows that the micromechanical model with imperfect interface can predict pretty well the properties of the rock as a function of porosity. Moreover, a sensitivity study results in a reference set of parameters for the ooid, the sparite matrix, and the interface for various oolitic limestones.

The proposed analytical framework is intended for materials with similar microstructures, for instance cement-based materials, carbonate rocks. The proposed model limitation consists in the assumption of concentric layered structure (i.e., perfectly spherical assemblage), isotropic phases, and linear interface behavior. The extension to an n-layered ellipsoidal assemblage is an ongoing study to account the anisotropy of the homogenized poroelastic properties. Moreover, considering non-linear behavior of interface also allows widening the applicability of the present model.

Appendix: Derivation of the bulk and shear moduli of n-layered porous inclusion based composites with imperfect interfaces

A.1. Effective bulk modulus

Considering the subproblem 1 described in Eq. (6). The medium is prescribed by a far field isotropic strain to determine the effective bulk modulus.

$$\xi_i(x)_{x \rightarrow \infty} = E^{iso} x_i \quad (52)$$

In spherical coordinates

$$\xi_r(r)_{r \rightarrow \infty} = E^{iso} r, \xi_\theta = 0, \xi_\varphi = 0 \quad (53)$$

According to Hervé and Zaoui (Hervé and Zaoui 1993), the non-zero displacement component and corresponding stress in phase i are given by

$$\begin{aligned} \xi_r^i &= H_i r + \frac{L_i}{r^2} \\ \sigma_{rr}^i &= 3H_i k_i - \frac{4L_i \mu_i}{r^3} \end{aligned} \quad (54)$$

where (H_i, L_i) are constants and (k_i, μ_i) are respectively the bulk modulus, shear modulus of phase i . The coefficient L_i must vanish in order to avoid a singularity at the origin, and $H_{n+1} = E^{iso}$, $L_{n+1} = 0$ according to the far field condition (53). The interface conditions described in Eqs. (3) and (4) at Γ_i may be written in the form

$$\begin{cases} \xi_r^{n+1}(r_n) = \xi_r^n(r_n) \text{ at interface } \Gamma_n \\ \sigma_{rr}^{n+1}(r_n) = \sigma_{rr}^n(r_n) \end{cases} \quad (55)$$

$$\begin{cases} \xi_r^{i+1}(r_i) = \xi_r^i(r_i) + \frac{\sigma_{rr}^i(r_i)}{D_r^i} \text{ at interface } \Gamma_i \text{ with } i = (1, n-1) \\ \sigma_{rr}^{i+1}(r_i) = \sigma_{rr}^i(r_i) \end{cases} \quad (56)$$

or in matrix form

$$\begin{aligned} I_{n+1}(r_n) \cdot V_{n+1} &= I_n(r_n) \cdot V_n \\ I_{i+1}(r_i) \cdot V_{i+1} &= [I_i(r_i) + J_i(r_i)] \cdot V_i \text{ with } i = (1, n-1) \end{aligned} \quad (57)$$

where

$$I_i(r) = \begin{bmatrix} r & \frac{1}{r^2} \\ 3k_i & -4\frac{\mu_i}{r^3} \end{bmatrix}, J_i(r) = \begin{bmatrix} \frac{3k_i}{D_r^i} & -4\frac{\mu_i}{D_r^i r^3} \\ 0 & 0 \end{bmatrix}, V_i = \begin{bmatrix} H_i \\ L_i \end{bmatrix} \quad (58)$$

The Eq. (57) leads to

$$\begin{aligned} V_{n+1} &= N_n \cdot V_n \\ V_n &= [N_{n-1} \dots N_2 \cdot N_1] \cdot V_1 = \left(\prod_{i=1}^{n-1} N_i \right) \cdot V_1 \end{aligned} \quad (59)$$

with

$$\begin{aligned} N_n &= [I_{n+1}(r_n)]^{-1} \cdot I_n(r_n) \\ N_i &= [I_{i+1}(r_i)]^{-1} \cdot [I_i(r_i) + J_i(r_i)] \text{ with } i = (1, n-1) \end{aligned} \quad (60)$$

The equilibrium condition at $r = r_n$ results in

$$V_{n+1} = N_n \cdot \left(\prod_{i=1}^{n-1} N_i \right) \cdot V_1 = N_n \cdot Q_{n-1} \cdot V_1 \quad (61)$$

With

$$\begin{aligned} N_n &= \frac{1}{3k_{n+1} + 4\mu_{n+1}} \begin{bmatrix} 3k_n + 4\mu_{n+1} & \frac{4}{r_n^3}(\mu_{n+1} - \mu_n) \\ 3(k_{n+1} - k_n)r_n^3 & 3k_{n+1} + 4\mu_n \end{bmatrix}, \\ Q_{n-1} &= \prod_{i=1}^{n-1} N_i = \begin{bmatrix} Q_{11}^{n-1} & Q_{12}^{n-1} \\ Q_{21}^{n-1} & Q_{22}^{n-1} \end{bmatrix} \\ V_{n+1} &= \begin{bmatrix} E^{iso} \\ 0 \end{bmatrix}, V_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix} \end{aligned} \quad (62)$$

It can be noticed that the unknown k_{n+1} only appears in the matrix N_n . The determined matrix Q_{n-1} denotes the interface conditions at Γ_i with $i = (1, n-1)$ in which the properties of each phase are well known. The relation (61) can be written in the form:

$$\begin{bmatrix} E^{iso} \\ 0 \end{bmatrix} = \frac{H_1}{3k_{n+1} + 4\mu_{n+1}} \begin{bmatrix} (3k_n + 4\mu_{n+1})Q_{11}^{n-1} + \frac{4}{r_n^3}(\mu_{n+1} - \mu_n)Q_{21}^{n-1} \\ 3(k_{n+1} - k_n)r_n^3Q_{11}^{n-1} + (3k_{n+1} + 4\mu_n)Q_{21}^{n-1} \end{bmatrix} \quad (63)$$

Finally, the effective bulk modulus k_{n+1} is given by

$$k_{n+1} = \frac{3k_n r_n^3 Q_{11}^{n-1} - 4\mu_n Q_{21}^{n-1}}{3(r_n^3 Q_{11}^{n-1} + Q_{21}^{n-1})} \quad (64)$$

A.2. Effective shear modulus

The effective shear modulus is derived by imposing a uniform deviatoric strain at the infinity

$$\xi_i(x)_{x \rightarrow \infty} = E_{ij}^{dev} x_j \quad (65)$$

In cartesian coordinate, the boundary displacement may be taken in the form (see Hashin and Monteiro (Hashin and Monteiro 2002))

$$\xi_{x1}^0 = -\beta x_1, \xi_{x2}^0 = -\beta x_2, \xi_{x3}^0 = 2\beta x_3 \quad (66)$$

i.e., $E_{11}^{dev} = -\beta$; $E_{33}^{dev} = -\beta$; $E_{33}^{dev} = 2\beta$; $E_{ij}^{dev} = 0$ with $i \neq j$ and β a constant. The solution of displacement and stress field of such problem in the spherical coordinate system was given by Hashin and Monteiro (Hashin and Monteiro 2002) as follows

$$\begin{aligned}\xi_r^i &= \beta \Xi_r^i(r) [3\cos^2(\theta) - 1], \xi_\theta^i = \beta \Xi_\theta^i(r) [-3\sin(\theta)\cos(\theta)] \\ \sigma_{rr}^i &= \beta \Sigma_{rr}^i(r) [3\cos^2(\theta) - 1], \sigma_{r\theta}^i = \beta \Sigma_{r\theta}^i(r) [-3\sin(\theta)\cos(\theta)]\end{aligned}\quad (67)$$

where $\Xi_r^i(r)$, $\Xi_\theta^i(r)$ and $\Sigma_{rr}^i(r)$, $\Sigma_{r\theta}^i(r)$ are only functions of r

$$\begin{cases} \Xi_r^i(r) = A_i r + B_i r^3 + \frac{C_i}{r^4} + \frac{D_i}{r^5} \\ \Xi_\theta^i(r) = A_i r + \frac{7-4\nu}{6\nu} B_i r^3 - \frac{2C_i}{3r^4} + \frac{2(1-2\nu)}{5-4\nu} \frac{D_i}{r^5} \\ \Sigma_{rr}^i(r) = 2\mu A_i - \mu B_i r^2 - \frac{8\mu C_i}{r^5} + \frac{4(5-\nu)}{3r^5} \frac{D_i}{r^3} \\ \Sigma_{r\theta}^i(r) = 2\mu A_i + \frac{7+2\nu}{3\nu} \mu B_i r^2 + \frac{16\mu C_i}{3r^5} + \frac{4(\nu+1)}{5-4\nu} \frac{D_i}{r^3} \end{cases}\quad (68)$$

where ν is the Poisson's ratio and A_i, B_i, C_i, D_i are unknown constants for layer i . The requirements that the displacement must be bounded at $r=0$ and $r \rightarrow \infty$, and that the stress must be also bounded at $r=0$ result in $C_1 = D_1 = B_{n+1} = 0$ and $A_{n+1} = \beta$. Christensen and Lo (Christensen and Lo 1979) showed $D_{n+1}=0$ for the case of a perfect interface and Hashin (Hashin 1991) also obtained $D_{n+1} = 0$ for the case of interface imperfect. Other constants A_i, B_i, C_i, D_i can be determined from the interface law described by relations (3) and (4), which can be written in the following form

$$\begin{cases} \xi_r^{n+1}(r_n) = \xi_r^n(r_n) \\ \xi_\theta^{n+1}(r_n) = \xi_\theta^n(r_n) \\ \sigma_{rr}^{n+1}(r_n) = \sigma_{rr}^n(r_n) \\ \sigma_{r\theta}^{n+1}(r_n) = \sigma_{r\theta}^n(r_n) \end{cases} \text{ at interface } \Gamma_n \quad (69)$$

$$\begin{cases} \xi_r^{i+1}(r_i) = \xi_r^i(r_i) + \frac{\sigma_{rr}^i(r_i)}{D_i^r} \\ \xi_\theta^{i+1}(r_i) = \xi_\theta^i(r_i) + \frac{\sigma_{r\theta}^i(r_i)}{D_i^\theta} \\ \sigma_{rr}^{i+1}(r_i) = \sigma_{rr}^i(r_i) \\ \sigma_{r\theta}^{i+1}(r_i) = \sigma_{r\theta}^i(r_i) \end{cases} \text{ at interface } \Gamma_i \text{ with } i = (1, n-1) \quad (70)$$

and in matrix form

$$\begin{aligned}X_{n+1}(r_n) \cdot W_{n+1} &= X_n(r_n) \cdot W_n \\ X_{i+1}(r_i) \cdot W_{i+1} &= [X_i(r_i) + Y_i(r_i)] \cdot W_i \text{ with } i = (1, n-1)\end{aligned}\quad (71)$$

where

$$X_i(r) = \begin{bmatrix} r & r^3 & \frac{1}{r^4} & \frac{1}{r^5} \\ r & \frac{7-4\nu_i}{6\nu_i} r^3 & -\frac{2}{3r^4} & \frac{1-2\nu_i}{5-4\nu_i} \frac{2}{r^5} \\ 2\mu_i & -\mu_i r^2 & -\frac{8}{r^5} \mu_i & \frac{20-4\nu_i}{4\nu_i-5} \frac{\mu_i}{r^3} \\ 2\mu_i & \frac{7+2\nu_i}{3\nu_i} \mu_i r^2 & \frac{16}{3r^5} \mu_i & \frac{4(\nu_i+1)}{5-4\nu_i} \frac{\mu_i}{r^3} \end{bmatrix} \quad (72)$$

$$Y_i(r) = \begin{bmatrix} \frac{2\mu_i}{D_r^i} & -\frac{\mu_i r^2}{D_r^i} & -\frac{8}{D_r^i r^5} \mu_i & \frac{20-4\nu_i}{4\nu_i-5} \frac{\mu_i}{D_r^i r^3} \\ \frac{2\mu_i}{D_\theta^i} & \frac{7+2\nu_i}{3\nu_i} \frac{\mu_i r^2}{D_\theta^i} & \frac{16}{3D_\theta^i r^5} \mu_i & \frac{4(\nu_i+1)}{5-4\nu_i} \frac{\mu_i}{D_\theta^i r^3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (73)$$

$$W_i = [A_i \ B_i \ C_i \ D_i]^T \quad (74)$$

Therefore

$$\begin{aligned}W_{n+1} &= M_n \cdot W_n \\ W_n &= [M_{n-1} \dots M_2 \cdot M_1] \cdot W_1 = \left(\prod_{i=1}^{n-1} M_i \right) \cdot W_1\end{aligned}\quad (75)$$

with

$$\begin{aligned}M_n &= [X_{n+1}(r_n)]^{-1} \cdot X_n(r_n) \\ M_i &= [X_{i+1}(r_i)]^{-1} \cdot [X_i(r_i) + Y_i(r_i)] \text{ with } i = (1, n-1)\end{aligned}\quad (76)$$

The equilibrium condition at $r = r_n$ is given by relation

$$W_{n+1} = M_n \cdot \left(\prod_{i=1}^{n-1} M_i \right) \cdot W_1 = M_n \cdot P_{n-1} \cdot W_1 = P_n \cdot W_1 \quad (77)$$

with

$$\begin{aligned}P_{n-1} &= \prod_{i=1}^{n-1} M_i \\ P_n &= M_n \cdot P_{n-1}\end{aligned}\quad (78)$$

$$\begin{aligned}W_1 &= [A_1 \ B_1 \ 0 \ 0]^T \\ W_{n+1} &= [A_{n+1} \ 0 \ C_{n+1} \ 0]^T\end{aligned}\quad (79)$$

It can be noticed that the unknown μ_{n+1} only appears in the matrix M_n . The matrix $P_n = M_n \cdot P_{n-1}$ only depends on radius r_i , the properties of phase i and interface Γ_i with $i = (1, n-1)$. The relation (77) can be written in the form

$$\begin{bmatrix} A_{n+1} \\ 0 \\ C_{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} P_{11}^n & P_{12}^n & P_{13}^n & P_{14}^n \\ P_{21}^n & P_{22}^n & P_{23}^n & P_{24}^n \\ P_{31}^n & P_{32}^n & P_{33}^n & P_{34}^n \\ P_{41}^n & P_{42}^n & P_{43}^n & P_{44}^n \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \\ 0 \\ 0 \end{bmatrix} \quad (80)$$

and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P_{21}^n & P_{22}^n \\ P_{41}^n & P_{42}^n \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \quad (81)$$

Equation (81) has solutions if

$$\text{Det} = \begin{vmatrix} P_{21}^n & P_{22}^n \\ P_{41}^n & P_{42}^n \end{vmatrix} = P_{21}^n P_{42}^n - P_{41}^n P_{22}^n = 0 \quad (82)$$

Combination Eq. (82) and $P_n = M_n \cdot P_{n-1}$ yields

$$\begin{aligned}
& (M_{22}^n M_{41}^n - M_{21}^n M_{42}^n) Z_{21} + (M_{23}^n M_{41}^n - M_{21}^n M_{43}^n) Z_{31} \\
& + (M_{22}^n M_{43}^n - M_{23}^n M_{42}^n) Z_{23} + (M_{21}^n M_{44}^n - M_{24}^n M_{41}^n) Z_{14} \\
& + (M_{22}^n M_{44}^n - M_{24}^n M_{42}^n) Z_{24} + (M_{23}^n M_{44}^n - M_{24}^n M_{43}^n) Z_{34} = 0
\end{aligned} \quad (83)$$

where $Z_{kl} = P_{k1}^{n-1} P_{l2}^{n-1} - P_{k2}^{n-1} P_{l1}^{n-1}$ with $k, l \in [1, 4]$ and M_{ij}^n, P_{ij}^{n-1} with $i, j \in [1, 4]$ are respectively the components of the matrix M_n and P_{n-1} defined by relations (76) and (78). Finally, by detailing the unknown matrix M_n , the effective shear modulus μ_{n+1} is the positive root of the following second order equation:

$$A \left(\frac{\mu_{n+1}}{\mu_n} \right)^2 + B \left(\frac{\mu_{n+1}}{\mu_n} \right) + C = 0 \quad (84)$$

with

$$\begin{aligned}
A &= A_1 r_n^{10} + A_2 r_n^7 + A_3 r_n^5 + A_4 r_n^3 + A_5 \\
A_1 &= 4Z_{21} (4v_n - 5) (10v_n - 7) \\
A_2 &= 20Z_{24} (8v_n^2 - 12v_n + 7) \\
A_3 &= 28Z_{23} (5 - 4v_n) + 72Z_{14} v_n \\
A_4 &= 40Z_{31} v_n (4v_n - 5) \\
A_5 &= 32Z_{34} v_n (5v_n - 4)
\end{aligned} \quad (85)$$

$$\begin{aligned}
B &= B_1 r_n^{10} + B_2 r_n^7 + B_3 r_n^5 + B_4 r_n^3 + B_5 \\
B_1 &= 3Z_{21} (4v_n - 5) (7 - 15v_n) \\
B_2 &= 60Z_{24} v_n (v_n - 3) \\
B_3 &= 56Z_{23} (4v_n - 5) - 144Z_{14} v_n \\
B_4 &= 80Z_{31} v_n (5 - 4v_n) \\
B_5 &= 16Z_{34} v_n (1 - 5v_n)
\end{aligned} \quad (86)$$

$$\begin{aligned}
C &= C_1 r_n^{10} + C_2 r_n^7 + C_3 r_n^5 + C_4 r_n^3 + C_5 \\
C_1 &= Z_{21} (4v_n - 5) (7 + 5v_n) \\
C_2 &= 10Z_{24} (7 - v_n^2) \\
C_3 &= 28Z_{23} (5 - 4v_n) + 72Z_{14} v_n \\
C_4 &= 40Z_{31} v_n (4v_n - 5) \\
C_5 &= 16Z_{34} v_n (7 - 5v_n)
\end{aligned} \quad (87)$$

Author Contributions Thi Thu Nga NGUYEN: Conceptualization; Formal analysis; Investigation; Methodology; Software; Validation; Visualization; Writing—original draft. Ngoc Bien NGUYEN: Conceptualization; Formal analysis; Investigation; Methodology; Software; Validation; Visualization. Minh Ngoc VU: Conceptualization; Formal analysis; Investigation; Methodology; Supervision; Validation; Visualization; Writing—original draft; and Writing—review & editing. Tuan NGUYEN-SY: Formal analysis; Methodology; Writing—review & editing. Duc Tho PHAM: Formal analysis; Methodology; Writing—review & editing. Nam Hung TRAN: Formal analysis; Methodology; Software; Writing—review & editing.

Data Availability The data used during the current study are available from the first author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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